AN INTRODUCTION TO FIXED POINT THEORY

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1. Introduction

Suppose we have some set X. Furthermore, suppose, for some $H \subseteq X$, we have a function $f : H \to X$. There are several things about the function that we might want to know:

- Is it continuous?
- Is it injective or surjective?
- Does it have a well-defined inverse?
- What properties does it preserve?
- Is $H \cap f(H)$ non-empty?

The last question is easy to answer if $H = X$, as we have $f(H) \subseteq H$, so $f(H) \cap H = f(H) \neq \emptyset$.

So, let us consider a dynamical system $f : H \to H$. Since, for all $h \in H$, $f(h) \in H$, we know that $f(f(h)) = f^2(h) \in H$. We can see inductively that for all $n \in \mathbb{N}$, that $f(f^n(h)) = f^{n+1}(h) \in H$. Note that $f^n(h)$ is the n^{th} iteration of f on h, and thus is not the same as $(f(h))^n$.

Definition 1. *Let $f : H \to H$, $a_0 \in H$, and for all $n \in \mathbb{N}$, let $a_n = f(a_{n-1})$. Then the sequence $a_0, a_1, a_2, ...$ is the* ***orbit*** *of a_0 under f. If, for some $m \in \mathbb{N}$, $f^m(a_0) = a_0$, then the orbit of a_0 under f is* ***periodic****, with period m, and a_0 is a* ***periodic point*** *of f.*

Now, understanding the behaviors of the orbits of the elements of the set is vital to understanding our system. We could work to find out if any of the orbits converge, or if they are periodic. One very important case of a periodic orbit to look for in our dynamical system is if there are any orbits of period 1. If the orbit of $a_0 \in H$ under f has period 1, then a_0 is a **fixed point** of f. In other words, a fixed point of f is simply an element $a_0 \in H$, such that $a_1 = f(a_0) = a_0$, and thus, for all $n \in \mathbb{N}$, $a_n = f^n(a_0) = a_0$.

Now, for certain systems, determining if there are fixed point, and what those fixed points are, is a rather simple matter, and can be done computationally. For instance, if $H = \mathbb{R}$, then

- if $f(x) = x$, then all the real numbers are fixed points of f, since $x = x$.
- if $f(x) = x + 1$, then there are no fixed points of f, since $0 \neq 1$.
- if $f(x) = x^2$, then 0 and 1 are the only fixed points of f, as $0 = x^2 - x$ only if $x = 0$ or $x = 1$.

However, if we have a more complicated system, such as $f : \mathbb{R} \to \mathbb{R}$, defined by $f(x) = \frac{e^{4x}}{x^2+x+1} + x^{\frac{2}{3}}$, or a function defined on the surface of a torus, it becomes much more difficult or perhaps even impossible to determine if there are are any fixed points and what they are through computation alone.

Fortunately, there are other ways of determine if a system has fixed points. For instance, though this is not the original theorem by Sharkovsky this is an immediate consequence

Theorem 1 (Sharkovsky)**.** *Suppose $f : \mathbb{R} \to \mathbb{R}$ is continuous. If f has a periodic periodic point, then f has a fixed point.*

The proof for this theorem can be found in [1].

Now, this theorem becomes very useful in cases where it is easier to find a real number with a period greater than one computationally than it is to find a fixed point. Note that this theorem does not tell us anything other than if there is a fixed point.

This theorem is not enough to determine if any given dynamical system has a fixed point. What if our set isn't $\mathbb{R}$? What if our function isn't necessarily continuous? What if we can't determine computationally if there are periodic points? Clearly, we need more theory to determine if a given dynamical system has a fixed point.

Fixed Point Theory is exactly what it sounds like: the study of determining if a dynamical system has a fixed point and what properties they might have. Fixed Point Theory intersects with several other fields in mathematics, and just as dynamical systems can vary greatly between fields, the theorems related to determining their fixed points each have their own conditions and conclusion. In this paper, we will look at four different major theorems in Fixed Point Theory, each of which applies to a certain kind of Dynamical System, and discuss some of their consequences.

2. Contractions on Metric Spaces

The work in this section draws heavily on [2].

2.1. Banach Contraction Theorem.

Let us begin by specifying that in this paper $\mathbb{R}^+ = \{x \in \mathbb{R} \mid x > 0\}$ and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

Definition 2. *Let E be a nonempty set. A **metric** on E is a function $d : E \to [0, \infty)$ with the following properties, for all $x, y, z \in E$:*

1. $d(x, y) = 0$ *if and only if* $x = y$,
2. $d(x, y) = d(y, x)$,
3. $d(x, y) + d(y, z) \geq d(x, z)$.

*The last property is known as the **triangle inequality**. A **metric space** is a set E with a metric. A metric space E with metric d can be written as (E, d).*

A metric is a way to describe the "distance" between elements of a set.

For example, let $E = \mathbb{R}^2$, and define $d : \mathbb{R}^2 \to \mathbb{R}^+$ such that for any $x = (x_1, x_2)$ and $y = (y_1, y_2)$ in $\mathbb{R}^2$, that

$$d(x, y) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}.$$

Now, we check if d is a metric. Let $x, y, z \in \mathbb{R}^2$, and then we have:

1. $d(x, y) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}$. If $x = y$, then $x_1 = y_1$ and $x_2 = y_2$, so $d(x, y) = 0$. If $x \neq y$, then $x_1 \neq y_1$ or $x_2 \neq y_2$, and thus $d(x, y) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2} > 0$.
2. $d(x, y) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2} = \sqrt{(y_1 - x_1)^2 + (y_2 - x_2)^2} = d(y, x)$
3. To get the triangle inequality, we recognize that for any $a, b, c \in \mathbb{R}^2$,

$$\begin{aligned} d(a - c, b - c) &= \sqrt{([a_1 - c_1] - [b_1 - c_1])^2 + ([a_2 - c_2] - [b_2 - c_2])^2} \\ &= \sqrt{(a_1 - b_1)^2 + (a_2 - b_2)^2} \\ &= d(a, b). \end{aligned}$$

So, for any $p, r \in \mathbb{R}^2$, and $\theta = (0, 0)$, if

$$d(p, \theta) + d(\theta, r) \geq d(p, r),$$

then for any $q \in \mathbb{R}^2$,

$$d(p + q, q) + d(q, r + q) \geq d(p, r).$$

We find that indeed, for every $p, r \in \mathbb{R}^2$, $d(p, \theta) + d(\theta, r) \geq d(p, r)$ as follows:

$$\begin{aligned}
(p_2 r_1 - p_1 r_2)^2 &\geq 0 \\
p_2^2 r_1^2 - 2p_1 p_2 r_1 r_2 + p_1^2 r_2^2 &\geq 0 \\
p_2^2 r_1^2 + p_1^2 r_2^2 &\geq 2p_1 p_2 r_1 r_2 \\
p_1^2 r_1^2 + p_2^2 r_1^2 + p_1^2 r_2^2 + p_2^2 r_2^2 &\geq p_1^2 r_1^2 + 2p_1 p_2 r_1 r_2 + p_2^2 r_2^2 \\
(p_1^2 + p_2^2) \cdot (r_1^2 + r_2^2) &\geq (-p_1 r_1 - p_2 r_2)^2 \\
2\sqrt{(p_1^2 + p_2^2) \cdot (r_1^2 + r_2^2)} &\geq -2p_1 r_1 - 2p_2 r_2 \\
p_1^2 + p_2^2 + 2\sqrt{(p_1^2 + p_2^2) \cdot (r_1^2 + r_2^2)} + r_1^2 + r_2^2 &\geq p_1^2 - 2p_1 r_1 + r_1^2 + p_2^2 - 2p_2 r_2 + r_2^2 \\
p_1^2 + p_2^2 + 2\sqrt{p_1^2 + p_2^2} \cdot \sqrt{r_1^2 + r_2^2} + r_1^2 + r_2^2 &\geq (p_1 - r_1)^2 + (p_2 - r_2)^2 \\
\left(\sqrt{p_1^2 + p_2^2} + \sqrt{r_1^2 + r_2^2}\right)^2 &\geq \left(\sqrt{(p_1 - r_1)^2 + (p_2 - r_2)^2}\right)^2 \\
\sqrt{(p_1)^2 + (p_2)^2} + \sqrt{(r_1)^2 + (r_2)^2} &\geq \sqrt{(p_1 - r_1)^2 + (p_2 - r_2)^2} \\
d(p, \theta) + d(\theta, r) &\geq d(p, r)
\end{aligned}$$

So, for any $x, y, z \in \mathbb{R}^2$, let $p = x - y$, $q = y$, and $r = z - y$. We know that

$$d(p, \theta) + d(\theta, r) \geq d(p, r),$$

which means that

$$d(p + q, q) + d(q, r + q) \geq d(p, r),$$

so we do indeed have

$$d(x, y) + d(y, z) \geq d(x, z).$$

So we have that the triangle inequality holds for d.

Thus, $(\mathbb{R}^2, d)$ is a metric space.

We can form a triangle from any three distinct points in $\mathbb{R}^2$. The sum of the lengths of any two sides of a triangle is always greater than or equal to the length of the third side and this is where the triangle inequality gets its name from.

For a second example, let E be a set, and for all $x, y \in E$, define

$$g(x, y) = \begin{cases} 1 & \text{if } x \neq y \\ 0 & \text{if } x = y. \end{cases}$$

For any $x, y, z \in E$ we have

(1) by definition $g(x, y) = 0$ if $x = y$, and $g(x, y) = 1 \neq 0$ if $x \neq y$,
(2) $g(x, y) = g(x, y)$, as both are 0 if $x = y$, and both are 1, if $y \neq x$.
(3) $g(x, y) + g(y, z) \geq g(x, z)$, because if $x = z$,

$$g(x, y) + g(y, z) \geq 0 = g(x, z),$$

and if $x \neq z$, then we cannot have $x = y$ and $y = z$, so

$$g(x, y) + g(y, z) \geq 1 = g(x, z).$$

Thus, g is indeed a metric on E. E was an arbitrary set though, and this metric can be used on any set. It is known as the **Discrete Metric**.

Now, before we get to our fixed point theorem, we must establish some definitions. For the following definitions, let (E, d) and (M, m) be metric spaces.

Definition 3. *A function $f : E \to M$ is* ***continuous*** *at $c \in E$ if for every $\epsilon > 0$, there exists some $\delta > 0$ such that, if $x \in E$, and $d(x, c) < \delta$, then $m(f(x), f(c)) < \epsilon$. If f is continuous at every $c \in E$, then f is* ***continuous*** *on E.*

Definition 4. *A function $f : E \to M$ is* ***uniformly continuous*** *if for every $\epsilon > 0$, there exists some $\delta > 0$ such that, for any $x, y \in E$, if $d(x, y) < \delta$, then $m(f(x), f(y)) < \epsilon$.*

We take a moment here to note that any function that is uniformly continuous is continuous.

Definition 5. *A function $f : E \to M$ is* ***Lipschitz continuous*** *if there exists a $\lambda \in \mathbb{R}^+$ such that for all $x, y \in E$,*

$$m(f(x), f(y)) \leq \lambda \cdot d(x, y).$$

For such a function, λ is called the ***Lipschitz constant****.*

A function that is Lipschitz continuous is uniformly continuous.

Definition 6. *A* ***Cauchy Sequence*** *in E is a sequence $x_0, x_1, ...,$ with all $x_i \in E$, such that $\lim_{\min(m,n)\to\infty} d(x_m, x_n) = 0$. A Cauchy sequence converges in E if there exists some $x \in E$ such that $\lim_{n\to\infty} x_n = x$.*

Definition 7. *E is* ***complete metric space*** *if every Cauchy sequence in E converges in E.*

The classic example of a complete metric space is $(\mathbb{R}, d)$, where $d(x, y) = |x - y|$ for all $x, y \in \mathbb{R}$. Though we will not go into the proof here, it can be found in [2].

However, the metric space $((0, 1), d)$, with $d(x, y) = |x - y|$ for all $x, y \in (0, 1)$, is not complete. Consider the sequence $a_1, a_2, ...,$ where $a_i = \frac{1}{n}$ for all $n \in \mathbb{N}$.

Definition 8. *A function $f : E \to E$ is a* ***contraction map*** *if it is Lipschitz continuous, with a Lipschitz constant less than one.*

Now, there are two last pieces of machinery we need before we can get to our theorem.

Proposition 1. *Let E and M be metric spaces. Then a function $f : E \to M$ is continuous at $p \in E$ if and only if, for every sequence of points $p_1, p_2, p_3, ...$ in E such that $\lim_{n\to\infty} p_n = p$, we have $\lim_{n\to\infty} f(p_n) = f(p)$.*

Proposition 2. *Any subsequence of a convergent sequence of points in a metric space converges to the same limit.*

The proofs of these are both given in [2].

Now, we have everything that we need to prove the Banach Contraction Theorem

Theorem 2 (Banach Contraction Theorem)**.** *Let E be a nonempty complete metric space, and $f : E \to E$ a contraction. Then there exists a unique point $p \in E$ such that $f(p) = p$. Furthermore, if $x \in E$ and then the orbit of x converges to p.*

Proof. Let E be a nonempty complete metric space with metric d. Let f be a contraction mapping, and $k \in [0, 1)$ such that for all $x, y \in E$,

$$d(f(x), f(y)) \leq kd(x, y).$$

Let $p_0 \in E$, and for $i \in \mathbb{N}$, let $p_i = f(p_{i-1})$. For any integer $n \in N_0$, we have

$$d(p_n, p_{n+1}) = d(f(p_{n-1}), f(p_n)) \leq kd(p_{n-1}, p_n).$$

We can repeat this process and we get $d(p_n, p_{n+1}) \leq kd(p_{n-1}, p_n) \leq k^2 d(p_{n-2}, p_{n-1}) \leq ... \leq k^r d(p_{n-r}, p_{n-r+1}) \leq ...$, which gives us

$$d(p_n, p_{n+1}) \leq k^n d(p_0, p_1).$$

Now, making use of the triangle inequality, we have, for any $n > m \geq 0$,

$$\begin{aligned} d(p_m, p_n) &\leq \sum_{i=m}^{n-1} d(p_i, p_{i+1}) \\ &\leq \sum_{i=m}^{n-1} k^i d(p_0, p_1) \\ &= k^m d(p_0, p_1) \sum_{i=0}^{n-m-1} k^i \\ &\leq k^m d(p_0, p_1) \sum_{i=0}^{\infty} k^i. \end{aligned}$$

Since $|k| < 1$, the series in the last line above is a convergent geometric series, and so we have

$$d(p_m, p_n) \leq \frac{k^m d(p_0, p_1)}{1 - k}.$$

Since $\lim_{m\to\infty} k^m = 0$, we have that $p_0, p_1, p_2, ...$ is a Cauchy sequence. E is a complete metric space, so (p_i) converges to a limit. Let $p = \lim_{i\to\infty} p_i$.

Now, since f is a contraction, it is Lipschitz continuous, and therefore continuous. Thus, from Proposition 1, we have

$$f(p) = \lim_{n\to\infty} f(p_n).$$

Since $p_1, p_2, p_3, ...$ is a subsequence of $p_0, p_1, p_2, ...$, we have, by Proposition 2, that

$$\lim_{n\to\infty} p_{n+1} = p.$$

Due to how we defined this sequence, we know that for all $n \in \mathbb{N}$, $f(p_n) = p_{n+1}$, and so we have

$$f(p) = \lim_{n\to\infty} f(p_n) = \lim_{n\to\infty} p_{n+1} = p,$$

so p is a fixed point of $f : E \to E$.

Now that we have a fixed point, we want to show that it is unique.

Suppose, indirectly, that p and q are both fixed points of f. Then

$$d(q, p) = d(f(q), f(p)) \leq kd(q, p).$$

Since $0 \leq k < 1$, we must have that $d(q, p) = 0$, which means that $q = p$. Thus, f has exactly one fixed point.

□

2.2. **Solutions to ODE's.** Mathematically modeling a real-world system can often results in some system of equations to describe its behavior. In many systems, what the system does over time can be described continuously, and what it does can dependent on what it is currently doing. For instance, the rate at which a swinging pendulum will accelerate at any given point in time is dependent upon its position and possibly its speed (depending upon what medium it is swinging through). What its speed will be in the next moment depends on its current speed and acceleration. There are many systems like this that are best modeled by differential equations. For this to be useful and reasonable, solutions need to exist. An **ordinary differential equation of order** n is an equation containing a function of one independent variable its derivatives, with n being the highest order derivative in the equation. A solution to an ordinary differential equation is a function that satisfies its initial conditions.

For instance, $cos(x)$ and e^x are both solutions to

$$\frac{d^4 f(x)}{dx^4} = f(x),$$

with the initial condition that $f(0) = 1$. This is a differential equation of order 4.

Through use of the Banach Contraction Theorem, we can show that solutions to first-order differential equations satisfying certain conditions exist and are unique. First, we must establish some definitions and theorems.

Definition 9. *For a metric space (E, d), let $f_1, f_2, ...$ be a sequence of functions defined on a set $A \subseteq E$. Then the sequence **converges uniformly** on A to a function $f : A \to E$ if, for every $\epsilon \in \mathbb{R}^+$, there exists some $N \in \mathbb{N}$ such that $d(f(x), f_n(x)) < \epsilon$ whenever $n \geq N$ and $x \in A$.*

Definition 10. *Let E be a metric space and G be a complete metric space with metric d. For $n \in \mathbb{N}$ let $f_n : E \to G$. Then the sequence of functions (f_n) is **uniformly convergent** if, for any $\epsilon \in \mathbb{R}^+$, there is a positive integer N such that if $n, m \geq N$ and are integers, then $d(f_n(p), f_m(p)) < \epsilon$ for all $p \in E$. If a function is uniformly convergent, it **uniformly converges** to some function $f : E \to G$, meaning that for every $\delta \in \mathbb{R}^+$*

Theorem 3. *Let E and G be metric spaces and $f_1, f_2, ...$ be a uniformly convergent sequence of continuous functions from E to G. Then $\lim_{n\to\infty} f_n$ is continuous.*

Though we do not provide proof here, the proof can be found in [2].

Definition 11. *For a metric space (E, d), for $x \in E$ and $r \in \mathbb{R}^+$, we define the r**-neighborhood** about x, $V_r(x)$, as the set $\{y \in E \mid d(x, y) < r\}$.*

Definition 12. *In a metric space E, $A \subseteq E$ is an **open set** if, for every $x \in A$, there exists some $r \in \mathbb{R}^+$ such that $B_r(x) \subseteq A$.*

Definition 13. *In a metric space E, let $A \subseteq E$. A **cover** of A is a (possibly infinite) collection of open sets $\{O_\lambda \mid \lambda \in \Lambda\}$ such that*

$$A \subseteq \bigcup_{\lambda \in \Lambda} O_\lambda.$$

Definition 14. *In a metric space E, $A \subseteq$ is **compact** if every cover of A has a finite subcover of A.*

Theorem 4. *Let E be a compact metric space, and G be a complete metric space with metric d. Then the set of all continuous functions from E to G is a complete metric space, with metric*

$$k(f, g) = \max\{d(f(p), g(p)) \mid p \in E\}.$$

A sequence of points of this metric space converges if and only if it is a uniformly convergent sequence of functions on E.

Proof. Let $\mathfrak{F}$ be the set of all continuous functions from E to G. The image of a compact set under a continuous function is compact, so for any $f, g \in \mathfrak{F}$, $f(E)$ and $g(E)$ are both compact. In addition, the product of compact sets is compact, so $g(E) \times f(E)$ is compact. Since any real-valued function of a compact metric space attains a maximum, we know that for any $f, g \in \mathfrak{F}$,

$$\max\{d(f(p), g(p)) \mid p \in E\}$$

exists. Thus, we may define, for any $f, g \in \mathfrak{F}$, the distance between f and g to be

$$k(f,g) = \max\{d(f(p), g(p) \mid p \in E\}.$$

Now, we want to show that $\mathfrak{F}$, with k, is a metric space.

Clearly, for all $f, g \in \mathfrak{F}$, $k(f,g) \geq 0$, and $k(f,g) = 0$ if and only if $f = g$.

We now establish the triangle inequality. Let $f, g, h \in \mathfrak{F}$ and let $p_0 \in E$ such the $k(f,h) = d(f(p_0), h(p_0))$. Then

$$\begin{aligned} k(f,h) &= d(f(p_0), h(p_0)) \\ &\leq d(f(p_0), g(p_0)) + d(g(p_0), h(p_0)) \\ &\leq \max\{d(f(p), g(p) \mid p \in E\} + \max\{d(g(p), h(p) \mid p \in E\} \\ &= k(f,g) + k(g,h). \end{aligned}$$

And so we do indeed have that $\mathfrak{F}$ is a metric space. It remains to be shown that $\mathfrak{F}$ is complete.

A sequence of points in $\mathfrak{F}$ is a sequence of continuous functions $f_1, f_2, f_3, \ldots$ from E to G. This sequence will converge to a point $f \in \mathfrak{F}$ if and only if, for every $\epsilon \in \mathbb{R}^+$, there is an $N \in \mathbb{N}$ such that for any integer $n \geq N$, we have

$$k(f, f_n) = \max\{d(f(p), f_n(p) \mid p \in E\} < \epsilon,$$

so

$$d(f(p), f_n(p)) < \epsilon$$

for all $p \in E$. Thus a sequence $f_1, f_2, \ldots$ in $\mathfrak{F}$ converges to $f \in \mathfrak{F}$ if and only if the sequence of functions $f_1, f_2, \ldots$ on E converges uniformly to the function f.

Now, suppose that a sequence of points $f_1, f_2, \ldots$ in $\mathfrak{F}$ is a Cauchy sequence in $\mathfrak{F}$. Then, for any $\epsilon \in \mathbb{R}^+$, there exists some $N \in \mathbb{N}$ such that for any integers $n, m \geq N$, we have

$$k(f_n, f_m) < \epsilon,$$

which gives us that

$$d(f_n(p), f_m(p)) < \epsilon$$

for all $p \in E$. From Proposition 1, we know that the sequence of functions must converge uniformly to some function $f : E \to G$. We then have, from Theorem 3, that f is continuous. Thus, every Cauchy sequence in $\mathfrak{F}$ converges in $\mathfrak{F}$, making it complete.

□

Theorem 5. *Let $U \subseteq \mathbb{R}^2$ be open, and let $(a,b) \in U$. If $f : U \to \mathbb{R}$ is Lipschitz continuous, then there exists $h \in \mathbb{R}^+$ such that there exists a unique function $\phi : (a-h, a+h) \to \mathbb{R}$ such that $\phi'(x) = f(x, \phi(x))$ for $x \in (a-h, a+h)$ and $\phi(a) = b$.*

Proof. If a function is differentiable on a set, then it is continuous on the set. So, if a ϕ like we claim exists, it is continuous on $(a-h, a+h)$. From the Fundamental Theorem of Caluclus, we then have, for all $x \in (a-h, a+h)$, that $\phi'(x) = f(x, \phi(x))$ and $\phi(a) = b$ if and only if

$$\phi(x) = \int_a^x f(t, \phi(t))dt + b.$$

Let d be the usual metric on $\mathbb{R}^2$ (the one used in the example), and let $\lambda \in \mathbb{R}^+$ such that for all $(x,y), (w,z) \in U$, $|f(x,y) - f(w,z)| \leq \lambda \cdot d((x,y), w, z))$.

We begin by choosing some $N \in \mathbb{R}^+$ such that $N > |f(a,b)|$. Let $\beta = N - |f(a,b)|$. Since f is Lipschitz continuous, and thus continuous, on U, and U is open, there exists some $r \in \mathbb{R}^+$ such that $B_r((a,b)) \subseteq U$ and if $d((a,b),(x,y)) < r$, then $|f(a,b) - f(x,y)| < \beta$, which means that $|f(x,y)| < N$.

Now, let $0 < h < \min\{\frac{r}{2}, \frac{r}{2N}, \frac{1}{M}\}$. Then we have that

$$R = \{(x,y) \in \mathbb{R}^2 \mid \ |x-a| \leq h, |y-b| < Nh\} \subseteq B_r((a,b)) \subseteq U.$$

So, we have that for any $(x, y) \in R$, $|f(x, y)| < N$.

Now, we want to prove that there exists a unique continuous function ϕ on the closed interval $[a-h, a+h]$ such that

$$\phi(x) = \int_a^x f(t, \phi(t))dt + b$$

for all $x \in [a-h, a+h]$.

Let $\mathfrak{F}$ be the set of all continuous functions that map from $[a-h, a+h]$, a compact metric space, to $\mathbb{R}$, a complete metric space, both with the usual absolute value metric. As we showed in Theorem 4, $\mathfrak{F}$ is a complete metric space with the metric k, which we define by

$$k(\psi, \omega) = \max\{|\psi(x) - \omega(x)| \mid x \in [a-h, a+h]\},$$

where $\psi, \omega \in \mathfrak{F}$.

Let B be the set of all continuous functions that map from $[a-h, a+h]$ to $[b-Nh, b+Nh]$ and map a to b. Since B is a closed subset of a complete metric space, B itself is a complete metric space with the same metric k.

Now, we want to show that if ϕ does exist as claimed, then $|\phi(x)-b| < Nh$ for all $x \in [a-h, a+h]$, which means that $\phi \in B$. Suppose that a function ϕ that we describe in our claim does exist, and there exist points $x \in [a-h, a+h]$ such that $|\phi(x)-b| \geq Nh$, let γ be the infimum of $|x-a|$ for all such points. Since ϕ is continuous, and $\phi(a) = b$, it follows that $\gamma > 0$ and $|\phi(a+\gamma) - b| = Nh$ or $|\phi(a-\gamma) - b| = Nh$. The cases are not particularly different, so, without loss of generality, assume $|\phi(a+\gamma) - b| = Nh$. By the Mean Value Theorem, we have that for some $\alpha \in (a, a+\gamma)$,

$$\frac{\phi(a+\gamma) - \phi(a)}{\gamma} = \phi'(\alpha).$$

From this we have

$$\begin{aligned} Nh &= |\phi(a+\gamma) - b| \\ &= |\phi(a+\gamma) - \phi(a)| \\ &= |\gamma\phi'(\alpha)| \\ &= |\gamma f(\alpha, \phi(\alpha))| \\ &< \gamma N \\ &\leq hN. \end{aligned}$$

This is a contradiction, so if ϕ does exist, then it is in B.

Now, for any $\psi \in B$, let us define a new function

$$F \circ \psi : [a-h, a+h] \to \mathbb{R}$$

by

$$(F \circ \psi)(x) = \int_a^x f(t, \psi(t))dt + b.$$

Since $\psi \in B$, we have that for all $t \in [a-h, a+h]$, $|\psi(t) - b| \leq Nh$. From this, we have that for all $t \in [a-h, a+h]$, $f(t, \psi(t))$ is defined, continuous as a function of t, and $|f(t, \psi(t))| < N$, since $(t, \psi(t)) \in R$. Since, f is a continuous function of t on $[a-h, a+h]$, we have that it is integrable on $[a-h, a+h]$.

Thus, for $x \in [a-h, a+h]$, $(F \circ \psi)(x)$ is defined, and

$$|(F \circ \psi)(x) - b| = |\int_a^x f(t, \psi(t))dt| \leq N|x-a| \leq Nh.$$

By the Fundamental Theorem of Calculus, we have that $F \circ \psi$ is continuous on $[a-h, a+h]$. Thus, we find that $F \circ \psi \in B$, which means that $F : B \to B$.

Now, we have chosen h to make F a contraction, as we will show. If $\psi, \omega \in B$, then for all $x \in [a-h, a+h]$ we have

$$\begin{aligned}
|(F \circ \psi)(x) - (F \circ \omega)(x)| &= \left| \left(\int_a^x f(t, \psi(t))dt + b \right) - \left(\int_a^x f(t, \omega(t))dt + b \right) \right| \\
&= \left| \int_a^x [f(t, \psi(t)) - f(t, \omega(t))]dt \right| \\
&\leq |x-a| \cdot \max\{|f(t,\psi(t)) - f(t,\omega(t))| \mid t \in [a-h, a+h]\} \\
&\leq |x-a| \cdot M \cdot \max\{|\psi(t) - \omega(t)| \mid t \in [a-h, a+h]\} \\
&\leq hM \cdot k(\psi, \omega).
\end{aligned}$$

Thus, we have

$$k(F \circ \psi, F \circ \omega) \leq hM \cdot k(\psi, \omega).$$

Since $hM < 1$, F is a contraction map. Therefore, Theorem 2 tells us that there exists a unique $\phi \in B$ such that $\phi = F \circ \phi$. In other words, there is a unique $\phi \in B$ such that for all $x \in [a-h, a+h]$

$$\phi(x) = \int_a^x f(t, \phi(t))dt + b.$$

And so we do indeed have that for $x \in (a-h, a+h)$, $\phi'(x) = f(x, \phi(x))$ and $\phi(a) = b$.

Thus we have that ϕ does exist, now we must prove that it is the only solution of our differential equation on $(a-h, a+h)$ with our initial condition. Note that the existence part of the proof, we could have replaced h with $h_1 \in \mathbb{R}^+$, with $h_1 < h$. Any solution of the differential equation on $(a-h, a+h)$ yields a solution to the integral equation on $[a-h_1, a+h_1]$. We know that there is a unique solution to the integral equation on $[a-h_1, a+h_1]$, which means that its solution must be the same as the solution to the integral equation on $[a-h, a+h]$. Since this is true for all h_1 such that $0 < h_1 < h$, there is at most one solution on $(a-h, a+h)$ and thus we have our claim.

□

Thus, given an equation of the form

$$\frac{d\psi}{dx}(x) = f(x, \psi(x))$$

with initial condition $\psi(a) = b$, where $(a, b) \in U$ and f is Lipschitz continuous, there exists a unique solution $\phi(x)$ on some interval $(a-h, a+h)$. If $U = \mathbb{R}^2$, then we can make h arbitrarily large and find that we have a unique solution to such a differential equation for all $x \in \mathbb{R}$.

Of course, the Banach Contraction Theorem has very specific requirements, and gives a very specific result. For determining that we have a complete metric space and a contraction, then we will have exactly one fixed point, and everything will converge to it. We know the eventual fate of our entire dynamical system. It is a nice luxury, but not always possible. Which is fine. We do not always need to know the exact number of fixed points, nor the exact destiny of every element of our system. Sometimes, it knowing that there is at least one fixed point is enough.

3. Isotone Functions on Posets

In this section, we draw primarily from [4].

3.1. **Knaster-Tarski Theorem.**

Definition 15. *A* ***partial order*** *is a binary operation $\preceq$ on a set P that is*

- ***reflexive:*** *$a \preceq a$, for all $a \in P$,*
- ***antisymmetric:*** *if $a \preceq b$ and $b \preceq a$, then $a = b$, for $a, b \in P$,*
- ***transitive:*** *for $a, b, c \in P$, if $a \preceq b$ and $b \preceq c$, then $a \preceq c$.*

*A set P with a partial ordering $\preceq$ is called a **partially ordered set**, or **poset**, denoted $(P, \preceq)$.*

Now, the most familiar example of a partially ordered set is $(\mathbb{R}, \leq)$. After a quick check, we can see that it satisfies all of the properties we need it to. However the definition of a poset does not require $a \preceq b$ or $b \preceq a$ for all $a, b \in P$.

Let $\mathbb{Z} \times \mathbb{Z}$ have the partial ordering $\preceq$, where $(a, b) \preceq (c, d)$ if and only if $a \leq c$ and $b \leq d$. If we check our properties, we find that this is indeed a poset

- $a \leq a$ and $b \leq b$, so $(a, b) \preceq (a, b)$.
- if $(a, b) \preceq (c, d)$ and $(c, d) \preceq (a, b)$, then $a \leq c \leq a$ so $a = c$ and $b \leq d \leq b$, so $b = d$, thus $(a, b) = c, d)$.
- if $(a, b) \preceq (c, d)$ and $(c, d) \preceq (e, f)$, then $a \leq c \leq e$, and $b \leq d \leq f$, so we do indeed have $(a, b) \preceq (e, f)$.

Now, something interesting to notice about this set is that not all of the elements are comparable. For instance $(-7, 9) \not\preceq (7, 8)$ and $(7, 8) \not\preceq (-7, 9)$.

Now, if we want to be able to compare all elements in the set with each other, then we must define another ordering.

Definition 16. *A **total ordering** on a set P is a partial ordering, $\preceq$, with the property that for all $a, b \in P$, $a \preceq b$ or $b \preceq a$. A set P with a total ordering $\preceq$ is called a **totally ordered set**, or a **chain**, and denoted $(P, \preceq)$.*

$(\mathbb{R}, \leq)$ is a totally ordered set.

Definition 17. *Let $(P, \preceq)$ be a partially ordered set and $M \subseteq P$ be nonempty. A map $f : P \to P$ is **isotone**, or **order-preserving**, if $f(x) \preceq f(y)$ whenever $x \preceq y$.*

Definition 18. *Let $(P, \preceq)$ be a partially ordered set. An element $x \in P$ is a **maximal** element if, for all $s \in P$, $x \preceq s$ implies $x = s$.*

Definition 19. *Let $(P, \preceq)$ be a partially ordered set. For any $C \subseteq P$, $u \in P$ is an **upper bound** if, for every $c \in C$, $c \preceq u$. Let*

$$U = \{u \in P \mid \forall c \in C, c \preceq u\},$$

*as in, let U be the set of all upper bounds of C. The **supremum** of C is the least upper bound of C. In other words, if v is the supremum of C, then $v \in U$, and for every $u \in U$ we have $v \preceq u$.*

Lemma 1 (Kuratowski-Zorn Lemma)**.** *If a partially ordered set P has the property that every chain has an upper bound in P, then the set P contains at least one maximal element.*

Theorem 6 (Knaster-Tarski Theorem)**.** *Let $(P, \preceq)$ be a poset and $f : P \to P$ be isotone. If there is a $b \in P$ such that $b \preceq f(b)$ and every chain in $A = \{x \in P \mid x \succeq b\}$ has a supremum. Then f has at least one fixed point, and at least one fixed point of f is maximal.*

Proof. Let P, A, f, and b be described as above. Now let us consider the partially ordered subset

$$Q = \{x \in P \mid x \preceq f(x)\} \cap \{x \in P \mid x \succeq b\}.$$

Clearly, $b \in Q$, so we know that Q is nonempty. By our hypotheses, every chain $C \subseteq Q$ has a supremum in A. So, for a given C, let $u = \sup(C)$. For each $c \in C$, $c \preceq u$, which means that $f(c) \preceq f(u)$, since f is isotone. By our construction of Q, $c \preceq f(c)$, and thus $c \preceq f(u)$. Since $c \preceq f(u)$ for all $c \in C$, $f(u)$ must be an upper bound of C, which means that $u \preceq f(u)$, and so $u \in Q$.

Since every chain of Q contains its supremum, we have, by the Kuratowski-Zorn Lemma, that there is a maximal element $\lambda \in Q$. Since $\lambda \in Q$, we have $\lambda \preceq f(\lambda)$, which, since f is isotone, gives us $f(\lambda) \preceq f(f(\lambda))$, so $f(\lambda) \in Q$. If $\lambda \prec f(\lambda)$, then this would contradict the maximality of λ. Thus, $\lambda = f(\lambda)$, so λ is a fixed point, and it is clearly maximal in P.

□

3.2. **Cantor-Schröder-Bernstein Theorem.** As a consequence of the Knaster-Tarski Theorem, we get the Cantor-Schröder-Bernstein Theorem, which a very useful theorem in determining if a bijection exists between sets, as an alternative to determining both an injection and a surjection from one set to the other.

Theorem 7 (Cantor-Schröder-Bernstein)**.** *Suppose that A and B are arbitrary sets. Then there is a bijection from A to B if and only if there is an injection from A to B and an injection from B to A.*

The proof of this theorem can be found in [6].

4. Continous Functions on Compact Convex Spaces

This section heavily uses work from [4].

4.1. **Brouwer's Fixed Point Theorem.**

Theorem 8 (Brouwer's Fixed Point Theorem)**.** *Let H be a compact, convex set with finite dimension, and $f : H \to H$ be continuous. Then f has a fixed point in H.*

Among fixed point theorems, Brouwer's Fixed Point Theorem is particularly well known. It is a topological result and has the potential to be used for continuous functions on finite dimensional compact convex sets in any topological space. For the scope of this paper, we will limit ourselves to a very specific kind of finite dimensional compact, convex sets in normed vector spaces, and then briefly discuss what could be done next. We note that a norm, $||\cdot||$, on a vector space E establishes a metric d on it, with $d(x, y) = ||x - y||$. Now, let us establish a few definitions to make the theorem clear.

Definition 20. *A set $A \subseteq E$ is **convex** if, for all $x, y \in A$ and $t \in [0, 1]$, $xt + y(t - 1) \in A$.*

A simple way to generally think about convex sets is that for any two points in the set, the line segment with those points as endpoint is contained in the set.

Now, to prove Brouwer's Theorem, we must first show that it works for at least one finite dimensional convex, compact set, for every number of dimensions. Then we can use that result to establish that it works for others.

4.2. **Simplices.** Let E be a normed vector space.

Definition 21. *For $s \in \mathbb{N}$, a finite set of $s + 1$ points $X = \{x_0, x_1, ..., x_s\} \subseteq E$ is **affinely independent** if the set $Y = \{x_1 - x_0, x_2 - x_0, ..., x_s - x_0\} \subseteq E$ is linearly independent.*

We note that for a set of $s + 1$ points to be affinely independent, we must have $s \leq \dim(E)$.

Consider the points $(2, 1), (1, 1), (-7, 7)$ in $\mathbb{R}^2$. Since $(-1, 0)$ and $(-9, 6)$ are linearly independent, our set of three points is affinely independent.

Definition 22. *Let $\{p_0, p_1, ..., p_s\}$ be an affinely independent set of $s+1$ points in E. Their convex hull*

$$\left\{ x \in E \,\middle|\, x = \sum_{i=0}^{s} \lambda_i p_i; 0 \leq \lambda_i \leq 1, \sum_{i=0}^{s} \lambda_i = 1 \right\}$$

*is called the (closed) **s-simplex** with vertices $p_0, ..., p_s$ and is denoted by $[p_0, p_1, ..., p_s]$, or σ^s if the vertices are not explicitly given, with s denoting the dimension of the simplex.*

A k-simplex $[q_0, q_1, ...q_k]$, where $\{q_0, q_1, ..., q_k\} \subset \{p_0, p_1, ..., p_s\}$, is called a ***k*-face** of the s-simplex $[p_0, p_1, ..., p_s]$. The **boundary**, $\partial\sigma^s$ is the union of all the faces of dimension less than or equal to $s-1$. The 0-faces are called the **vertices** of σ^s.

Using the same three points as before, we have that the 2-simplex $[(2,1),(1,1),(-7,7)]$ is the triangle in $\mathbb{R}^2$ formed by those three points.

In $\mathbb{R}^3$, a 0-simplex is a point, a 1-simplex is a line segment, a 2-simplex is a triangle, and a 3-simplex is a tetrahedron.

4.3. **Triangulation of $\mathbf{S}^n$.** Let E be the normed space of all sequences $x = (x_1, x_2, ...)$ of real numbers having finitely many nonzero x_n, with the norm

$$||x|| = \sum_0^\infty |x_i|.$$

The subset $\{x \in E \mid x_i = 0 \text{ for all } i > n\}$ is denoted by E^n. The (closed) *unit n-ball* is

$$\mathbf{K}^n = \{x \in E^n \mid \ ||x|| \leq 1\}$$

The *unit n-sphere* is

$$\mathbf{S}^n = \{x \in E^{n+1} \mid \ ||x|| = 1\}.$$

Its upper hemisphere is $\mathbf{S}^n_+ = \{x \in \mathbf{S}^n \mid \ x_{n+1} \geq 0\}$, and its lower hemisphere is $\mathbf{S}^n_- = \{x \in \mathbf{S}^n \mid x_{n+1} \leq 0\}$.

We note here that $\mathbf{S}^n = \mathbf{S}^n_+ \bigcup \mathbf{S}^n_-$, and that

$$\mathbf{S}^{n-1} = \{x \in \mathbf{S}^n \mid \ x_{n+1} = 0\} = \mathbf{S}^n_+ \bigcap \mathbf{S}^n_-.$$

More generally, for $k < n$, we have

$$\mathbf{S}^k = \{x \in \mathbf{S}^n \mid \ x_{k+2} = x_{k+3} = ... = x_{n+1} = 0\}.$$

Finally, we note that $\mathbf{S}^{n+1}$ is the boundary of $\mathbf{K}^n$.

Definition 23. *A finite family of simplices in $\mathbf{S}^n$, Ω^n, is called a **triangulation** of $\mathbf{S}^n$ if:*

1. *the intersection of any two simplices in Ω^n is either empty or a common face of each,*
2. *if $\sigma \in \Omega^n$ then every face of σ is also in Ω^n,*
3. *$\mathbf{S}^n = \bigcup\{\sigma \mid \sigma \in \Omega^n\}$,*
4. *each $(n-1)$-simplex of Ω^n is the common face of exactly two n-simplices in Ω^n.*

For any set $B \in E$, we define $\alpha : B \to E$ to be an **antipodal map** if for every $x \in B$, $\alpha(x) = -x$. For any $C \subseteq B$, C and $\alpha(C)$ are called **antipodal**. A map $f : B \to E$ is **antipode-preserving** if for all $x \in B$, $\alpha(f(x)) = f(\alpha(x))$.

Now a more specific kind of triangulation is necessary for some of our proofs:

Definition 24. *A triangulation Ω^n of $\mathbf{S}^n$ is **symmetric** if:*

1. *for every $k \leq n$, the k-sphere $\mathbf{S}^k$ is a union of k-simplices of Ω^n,*
2. *for every $k \leq n$, for each $\sigma^k \in \Omega^n$, $\alpha(\sigma^k)$ is also a k-simplex of Ω^n.*

Now, there is a specific symmetric triangulation of $\mathbf{S}^n$ that we still need to define. For $i \in \{0, 1, ..., n\}$, let $e_i = (\delta^i_0, \delta^i_1, \delta^i_2, ...) \in E$, where δ^i_j is the Kronecker delta. Clearly, the unit ball $\mathbf{K}^{n+1}$ is the convex hull of the set $\{e_0, e_1, ..., e_n, -e_0, -e_1, ..., -e_n\}$ and the set of all n-simplices $[\pm e_0, ..., \pm e_n]$ and all their faces provides a triangulation of $\mathbf{S}^n$, which we call the **basic triangulation** and denote by Σ^n. We note that each simplex in Σ^n has a unique representation (called its **standard form**) of the form $[\pm e_{i_0}, ..., \pm e_{i_s}]$, where $0 \leq i_0 < ... < i_s \leq n$.

4.4. A Combinatorial Helper.

Definition 25. *Let Ω^k be a triangulation of $\boldsymbol{S}^k$ and Ω^n be a triangulation of $\boldsymbol{S}^n$. A map f of the vertices of Ω^k to the vertices of Ω^n is called a* ***simplicial vertex map*** *if for each simplex $[p_0, ..., p_s]$ of Ω^k, the points $f(p_0), ..., f(p_s)$ are vertices of a (possibly lower-dimensional) simplex of Ω^n. Clearly, f extends to a map $\boldsymbol{S}^k \to \boldsymbol{S}^n$ (denoted also by f) sending simplices of Ω^k to simplices of Ω^n.*

Definition 26. *Let Ω^k be an arbitrary triangulation of $\boldsymbol{S}^k$, and let $f : \Omega^k \to \Sigma^n$ be a simplicial vertex map. An r-simplex $[p_0, ..., p_r]$ of Ω^k is called* ***positive****, with respect to f, if:*

1. *the vertices $f(p_0), ..., f(p_r)$ span an r-simplex $\sigma^r \in \Sigma^n$,*
2. *the standard form of σ^r is "alternating in sign",*
$$\sigma^r = [+e_{i_0}, -e_{i_1}, ..., (-1)^r e_{i_r}],$$
with the first vertex being positive.

An r-simplex $[p_0, ..., p_r]$ of Ω^k is called ***negative****, with respect to f, if:*

1. *the vertices $f(p_0), ..., f(p_r)$ span an r-simplex $\sigma^r \in \Sigma^n$,*
2. *the standard form of σ^r is "alternating in sign",*
$$\sigma^r = [-e_{i_0}, +e_{i_1}, ..., (-1)^{r+1} e_{i_r}],$$
with the first vertex being negative.

An r-simplex Ω^k is called ***neutral****, with respect to f, if it is not positive and not negative.*

For any simplicial vertex map $f : \Omega^k \to \Sigma^n$ and any $\mathbf{L} \subseteq \mathbf{S}^k$, the number of positive r-simplices in L under f is denoted by $p(f, \mathbf{L}, r)$.

Proposition 3. *Let $k \leq n$, and let $f : \Omega^k \to \Sigma^n$ be a simplicial vertex map of a symmetric triangulation of $\boldsymbol{S}^k$ into Σ^n. If f is antipode-preserving, then*
$$p(f, \boldsymbol{S}^k, k) \equiv p(f, \boldsymbol{S}^{k-1}, k-1) \mod (2).$$

Proof. Let us consider the upper hemisphere $\mathbf{S}^k_+$ of the k-sphere. We can decompose the set of k-simplices in $\mathbf{S}^k_+$ in to three disjoint classes:

$$A_+ = \{\sigma^k \subset \mathbf{S}^k_+ \mid \sigma^k \text{ is positive}\},$$
$$A_- = \{\sigma^k \subset \mathbf{S}^k_+ \mid \sigma^k \text{ is negative}\},$$
$$A_0 = \{\sigma^k \subset \mathbf{S}^k_+ \mid \sigma^k \text{ is neutral}\}.$$

Now, let
$$T = \sum_{\sigma^k \in A_+} p(f, \sigma^k, k-1) + \sum_{\sigma^k \in A_-} p(f, \sigma^k, k-1) + \sum_{\sigma^k \in A_0} p(f, \sigma^k, k-1).$$

Because each $p(f, \sigma^k, k-1)$ is the number of positive $(k-1)$-faces of σ^k, the sum T counts every positive σ^{k-1} in $\mathbf{S}^k_+$ at least once. Recall that by definition of a triangulation, any $\sigma^{k-1} \in \mathbf{S}^k$ is the common face of exactly two $\sigma^k \in \mathbf{S}^k$. Now, if both of the k-simplices for which σ^{k-1} is a face are in $\mathbf{S}^k_+$, and σ^{k-1} is positive, then clearly σ^{k-1} is twice in T. However, if only one of the k-simplices for which σ^{k-1} is a face is contained in $\mathbf{S}^k_+$, the other is contained in $\mathbf{S}^k_-$. If σ^{k-1} is positive, then σ^{k-1} is only counted once in T. If σ^{k-1} is a face of a k-simplex in $\mathbf{S}^k_+$ and a k-simplex in $\mathbf{S}^k_-$, then it is in $\mathbf{S}^{k-1}$. Thus we have that $T \equiv p(f, \mathbf{S}^{k-1}, k-1) \mod (2)$.

Let us again consider
$$T = \sum_{\sigma^k \in A_+} p(f, \sigma^k, k-1) + \sum_{\sigma^k \in A_-} p(f, \sigma^k, k-1) + \sum_{\sigma^k \in A_0} p(f, \sigma^k, k-1).$$

Suppose that σ^k is neutral. If f maps σ^k to a simplex of dimension less than $k-1$, then clearly, any $(k-1)$-face of σ^k could not be positive, as it would not be able to map to a $(k-1)$-simplex. Thus the only neutral σ^k that could be counted in T are those such that $f(\sigma^k) = [\pm e_{i_0}, ..., \pm e_{i_k}]$ where $i_0 \leq i_1 \leq ... \leq i_k$ and there is either exactly one repeated vertex, or all the vertices are distinct but their signs do not alternate (note that since this is a simplicial vertex map, e_i and $-e_i$ cannot both be in $f(\sigma^k)$). In either case, a positive $(k-1)$-face of σ^k can only occur if there is at most one pair of adjacent vertices, e_{i_j} and $e_{i_{j+1}}$, where $0 \leq j < k$, with the same sign in $f(\sigma^k)$. Clearly then, if the $(k-1)$-face of σ^k with $f^{-1}(e_{i_j})$ removed is positive, then so is the $(k-1)$-face of σ^k with $f^{-1}(e_{i_{j+1}})$ removed. Thus, $p(f, \sigma^k, k-1)$ is even for each $\sigma^k \in A_0$. From this, we have

$$T \equiv \sum_{\sigma^k \in A_+} p(f, \sigma^k, k-1) + \sum_{\sigma^k \in A_-} p(f, \sigma^k, k-1) \mod (2).$$

Now, if σ^k is positive, meaning that $f(\sigma^k) = [e_{i_0}, -e_{i_1}, ..., (-1)^k e_{i_k}]$, then each vertex of σ^k maps to a distinct vertex, meaning that each $(k-1)$-face maps to a distinct $(k-1)$-face of $f(\sigma^k)$. Thus, the only positive $(k-1)$-face of σ^k is one that f maps to $[e_{i_0}, -e_{i_1}, ..., (-1)^{k-1} e_{i_{k-1}}]$, because the removal of any vertex other than $(-1)^k e_{i_k}$ would result in the standard form not alternating in sign, or the first vertex would be negative. So, a positive σ^k has only one positive $(k-1)$-face. Following similar reasoning, we find that if σ^k is negative, then it only contains one positive $(k-1)$-face. Thus, we have that

$$\sum_{\sigma^k \in A_+} p(f, \sigma^k, k-1) = |A_+|, \text{ and } \sum_{\sigma^k \in A_-} p(f, \sigma^k, k-1) = |A_-|.$$

From this, we have

$$T \equiv |A_+| + |A_-| \mod (2).$$

Since f is antipode-preserving, an k-simplex $\sigma^k \subseteq \mathbf{S}^k_+$ is negative if and only if $\alpha(\sigma^k) \subseteq \mathbf{S}^k_-$ is positive. This means that $|A_-| = |\{\sigma^k \subseteq \mathbf{S}^k_- \mid \sigma^k \text{ is positive }\}|$. This gives us that $|A_+| + |A_-| = p(f, \mathbf{S}^k, k)$, which then in turn gives us

$$p(f, \mathbf{S}^k, k) \equiv |A_+| + |A_-| \equiv T \equiv p(f, \mathbf{S}^{k-1}, k-1) \mod (2).$$

□

Theorem 9. *Let $f : \Omega^n \to \Sigma^n$ be a simplicial vertex map of a symmetric triangulation of $\boldsymbol{S}^n$. If f is antipode-preserving, then f maps an odd number of simplices of $\boldsymbol{S}^n$ to*

$$\sigma^n_0 = [e_0, -e_1, ..., (-1)^n e_n].$$

Proof. By definition, an n-simplex $\sigma^n \in \Omega^n$ is positive if and only if $f(\sigma^n) = \sigma^n_0$. We have, by Proposition 3, that

$$p(f, \mathbf{S}^n, n) \equiv p(f, \mathbf{S}^{n-1}, n-1) \equiv ... \equiv p(f, \mathbf{S}^0, 0) \mod (2).$$

Since $\mathbf{S}^0$ is two antipodal vertices, and f maps them onto a pair of antipodal vertices, clearly one will be positive and the other negative, so $p(f, \mathbf{S}^0, 0) = 1$, which is odd, and thus we have out claim. □

4.5. **The Lusternik-Schnirelmann-Borsuk Theorem.** For some set $A \subseteq E$, we define the **diameter** of A, $\delta(A)$, as the maximum distance between any two points in A, as in

$$\delta(A) = \max(\{||x - y|| \mid x, y \in A\}).$$

Lemma 2 (Lebesgue)**.** *Let $\{M_1, ..., M_y\}$ be a family of closed non-empty sets in a compact metric space X such that $\bigcap_{i=1}^{y} M_i = \emptyset$. Then there exists a $\epsilon > 0$ with the property: any subset $A \subseteq X$ that intersects with every M_i must have $\delta(A) \geq \epsilon$.*

Proof. Let Z be the compact metric space $M_1 \times ... \times M_y$ and let $\kappa : Z \to \mathbb{R}$ be defined by

$$\kappa((x_1, ..., x_y)) = \max\{d(x_i, x_j) \mid 1 \leq i \leq j \leq y\}.$$

We take a moment to observe that κ is continuous. Because $\bigcap_{i=1}^{y} M_i = \emptyset$, it is impossible for each x_i to be the same, making the map κ always be non-zero. Thus it has some minimum value $\epsilon > 0$. If $A \subset X$ intersects each M_i, there is an $x_i \in A \bigcap M_i$ for each $i \in \{1, 2, ..., y\}$. Since $\kappa(x_1, ..., x_y) \geq \epsilon$, at least one $d(x_i, x_j) \geq \epsilon$, so $\delta(A) \geq \epsilon$.

□

Theorem 10 (Lebesgue)**.** *Let $\{M_1, ..., M_y\}$ be a closed covering of a compact metric space X. Then there exists a $\lambda > 0$ (a Lebesgue number of the covering) with the property: if there exists a set A such that $\delta(A) < \lambda$ and for all $j \in \{1, 2, ..., r\}$, where $r \leq y$, we have $A \bigcap M_{i_j} \neq \emptyset$, then*

$$\bigcap_{j=1}^{r} M_{i_j} \neq \emptyset.$$

Proof. Let $R = \{\{M_{i_1}, M_{i_2}, ..., M_{i_j}\} \mid \bigcap_{x=1}^{j} M_{i_x} = \emptyset\}$. In other words let R be the set of all subsets of our covering that have an empty intersection. By Lemma 2, every element $X \in R$ must have an ϵ such that if A intersects with all of the elements of X, then $\delta(A) \geq \epsilon$. So, let $lambda = \min\{\epsilon \mid \epsilon \text{ is the } \epsilon \text{ associated with } X \in R\}$. Now, suppose that A intersects with multiple elements of the covering, and $\delta(A) < \lambda$. If the elements of the covering had an empty intersection, then $\delta(A)$ would have to be greater than or equal to λ. Since this is not the case, they must have a nonempty intersection.

□

Lemma 3. *Let $M_1, ..., M_{n+1}$ be $n+1$ closed sets on $\boldsymbol{S}^n$, no one of which contains a pair of antipodal points. If the family*

$$\{M_1, M_2, ..., M_{n+1}, \alpha(M_1), \alpha(M_2), ..., \alpha(M_{n+1})\}$$

covers $\boldsymbol{S}^n$, then $M_1 \bigcap M_2 \bigcap ... \bigcap M_{n+1} \neq \emptyset$.

Proof. Let us denote $\alpha(M_i)$ as M_{-i}. Since M_i does not contain any pair of antipodal points, we have $d(M_i, M_{-i}) = \epsilon_i > 0$ for each $i = 1, ..., n+1$.

We linearly order the covering by

$$M_1, M_{-1}, M_2, M_{-2}, ..., M_{n+1}, M_{-n-1},$$

and let λ be a Lebesgue number for this closed covering.

Let Ω^n be a symmetric triangulation of $\mathbf{S}^n$, where the diameter of each simplex is less than $\epsilon = \min(\lambda, \epsilon_1, \epsilon_2, ..., \epsilon_{n+1})$. Note that depending upon the value of ϵ, Σ^n may not be an acceptable triangulation. We first construct a simplicial vertex map $f : \Omega^n \to \Sigma^n$ as follows:

For each vertex $p \in \Omega^n$, let M_j be the first set of the ordered covering containing p, and set

$$f(p) = \frac{j}{|j|}(-1)^{j+1} e_{|j|}.$$

We note that Σ^n can be described as the set of all simplices $[\pm e_{i_0}, \pm e_{i_1}, ..., \pm e_{i_s}]$, where $s \leq n$ and no simplex contains antipodal vertices. For any two vertices p_i, p_j of a simplex of $\sigma \in \Omega^n$, we have that if $p_i \in M_k$, that $p_j \notin M_{-k}$, as

$$d(p_i, p_j) < \epsilon \leq \epsilon_k = d(M_k, M_{-k}).$$

We can see from this that we cannot have $f(p_i) = (-1)^{k+1} e_{|k|}$ and $f(p_j) = (-1) \cdot (-1)^{1-k} e_{|k|} = -f(p_i)$. Since no two vertices of any simplex of Ω^n gets mapped to antipodal vertices in Σ^n and each vertex of Ω^n gets mapped to a vertex in Σ^n, f is indeed a simplicial vertex map.

If the first set to contain p_i is M_k, then clearly the first set to contain $-p_i$ is M_{-k}. Thus, $\alpha \circ f = f \circ \alpha$, so from Theorem 9, there is some simplex $[p_1, ..., p_{n+1}]$ such that

$$f[p_1, ..., p_{n+1}] = [e_1, -e_2, ..., (-1)^n e_{n+1}].$$

This means that each $p_i \in M_i$, so that $[p_1, ..., p_{n+1}] \bigcap M_i \neq \emptyset$ for $i = 1, ..., n+1$. Since $\delta([p_1, ..., p_{n+1}]) < \epsilon \leq \lambda$, it follows that $M_1 \bigcap ... \bigcap M_{n+1} \neq \emptyset$. Thus we have our claim.

□

Theorem 11 (Lusternik-Schnirelmann-Borsuk)**.** *In any closed covering $\{M_1, M_2, ..., M_{n+1}\}$ of $\boldsymbol{S}^n$ by $n+1$ sets, at least one set M_i must contain a pair of antipodal points.*

Proof. Assume, indirectly, that no M_i contains a pair of antipodal points. Then by Lemma 3, since $\{M_1, ..., M_{n+1}, \alpha(M_1), ..., \alpha(M_{n+1})\}$ would cover $\mathbf{S}^n$, we would have that $M_1 \bigcap ... \bigcap M_{n+1} \neq \emptyset$. Since $\{M_1, M_2, ..., M_{n+1}\}$ covers $\mathbf{S}^n$, that must mean that $\{\alpha(M_1), ..., \alpha(M_{n+1})\}$ must also cover $\mathbf{S}^n$. So, for any $x \in M_1 \bigcap ... \bigcap M_{n+1}$, there must be some $\alpha(M_j)$, for $1 \leq j \leq n+1$ such that $x \in \alpha(M_j)$. However, this means that $\alpha(x) \in M_j$. This would mean that M_j would contain a pair of antipodal points, which is a contradiction, and thus we have our claim.

□

4.6. **Borsuk's Antipodal Theorem.**

Definition 27. *Let X and Y be two normed vector spaces and $A \subseteq X$. A continuous map $f : A \to Y$ is called **extendable** over X if there is a continuous map $F : X \to Y$ such that $F|_A = f|_A$.*

Definition 28. *Two continuous maps $f : X \to Y$ and $g : X \to Y$ are called **homotopic** if there is a continuous map $H : X \times [0,1] \to Y$ such that $H(x,0) = f(x)$ and $H(x,1) = g(x)$ for all $x \in X$. The map H is called a **homotopy** of f to g and is written $H : f \simeq g$. For each $t \in [0,1]$, the map $x \mapsto H(x,t)$ is denoted by $H_t : X \to Y$.*

Clearly, the family $\{H_t : X \to Y\}_{0 \leq t \leq 1}$ determines H and vice versa. The relation of homotopy is an equivalence relation, so it is reflexive, symmetric, and transitive.

Definition 29. *If a continuous map $f : X \to Y$ is homotopic to a constant map, $g(x) = c$ for all $x \in X$, then we call f **nullhomotopic**, and write $f \simeq 0$.*

Theorem 12. *A continuous map $f : \boldsymbol{S}^n \to Y$ is nullhomotopic if and only if it is extendable to an $F : \boldsymbol{K}^{n+1} \to Y$.*

Proof. Assume that $f : \mathbf{S}^n \to Y$ is extendable to $F : \mathbf{K}^{n+1} \to Y$. For $x \in \mathbf{S}^n$ and $0 \leq t \leq 1$, let $H : \mathbf{K}^{n+1} \times \mathbf{I} \to Y$, where $H(x,t) = F(xt)$. Then we can see that $H : 0 \simeq f$.

Conversely, suppose that $H : \mathbf{S}^n \times \mathbf{I} \to Y$ shows that f is nullhomotopic, where, for all $x \in \mathbf{S}^n$, $H(x,1) = f(x)$ and $H(x,0) = c$, where c is some constant. Let us define $F : \mathbf{K}^{n+1} \to Y$ by

$$F(y) = \begin{cases} c, & 0 \leq ||y|| \leq \frac{1}{2} \\ H\left(\frac{y}{||y||}, 2||y|| - 1\right), & \frac{1}{2} \leq ||y|| \leq 1 \end{cases},$$

We can see that F is continuous, and that for $y \in \mathbf{S}^n$, we have $F(y) = f(y)$, thus F is an extension of f to $\mathbf{K}^{n+1}$.

□

Theorem 13. *There is no antipode-preserving continuous map $f : \boldsymbol{S}^{n+1} \to \boldsymbol{S}^n$.*

Proof. Suppose, indirectly, that $h : \mathbf{S}^{n+1} \to \mathbf{S}^n$ is continuous and antipode-preserving. Let $g : E \setminus \{0\} \to E$ be defined by $g(x) = \frac{x}{||x||}$. Let σ be an $(n+1)$-simplex centered at 0 and the n-faces of σ be $F_1, F_2, ..., F_{n+2}$. We note here that this means that no face of σ contains 0. We can decompose $\mathbf{S}^n$ into $n+2$ closed sets $A_1, A_2, ..., A_{n+2}$ by projecting the boundary of our simplex, $\partial\sigma$, onto $\mathbf{S}^n$, and defining our sets by $A_i = g(F_i)$. With a bit of work, one can see that $g : \partial\sigma \to \mathbf{S}^n$ is a homeomorphism.

Now suppose, indirectly, that some A_i contains a pair of antipodal points y and $\alpha(y)$. Without loss of generality, let us suppose that A_1 contains a pair of antipodal points. Since each face of $\sigma = [p_0, p_1, ..., p_{n+1}]$ is an n-simplex itself, let $F_i = [p_0, p_1, ..., p_n]$, and $x, z \in F_i$, where $g(x) = \alpha(g(z))$. Then, we have $\frac{x}{||x||} = \frac{-z}{||z||}$, so $x \cdot ||z|| = -z \cdot ||x||$. Now, since F_i is an n-simplex, let $x = \sum_{i=0}^{i=n} \lambda_i p_i$, where $0 \leq \lambda_i \leq 1$ and $\sum_{i=0}^{i=n} \lambda_i = 1$. Likewise, let $z = \sum_{i=0}^{i=n} \kappa_i p_i$, where $0 \leq \kappa_i \leq 1$ and $\sum_{i=0}^{i=n} \kappa_i = 1$. This gives us

$$||z|| \cdot \sum_{i=0}^{i=n} \lambda_i p_i = -||x|| \cdot \sum_{i=0}^{i=n} \kappa_i p_i.$$

From this we have

$$0 = \sum_{i=0}^{i=n} (||z||\lambda_i + ||x||\kappa_i) p_i.$$

Let $\beta_i = ||z||\lambda_i + ||x||\kappa_i$ and $\beta = \sum_{i=0}^{n} \beta_i$. Then, clearly we have

$$0 = \sum_{i=0}^{n} \frac{\beta_i}{\beta} p_i.$$

However, since $0 \leq \frac{\beta_i}{\beta} \leq 1$ and $\sum_{i=0}^{n} \frac{\beta_i}{\beta} = 1$, this means that $0 \in F_i$, which is a contradiction. Thus, no A_i contains a pair of antipodal points.

Now, let $M_i = h^{-1}(A_i)$, for $i \in \{1, 2, ..., n+2\}$. Since f is continuous, we have that each M_i is closed, and their union covers $\mathbf{S}^{n+1}$. So by the Lusternik-Schnirelmann-Borsuk Theorem, there exists some $x \in M_i \bigcap \alpha(M_i)$ for some i. Because f is antipode-preserving, this means that $f(x)$ and $f(\alpha(x)) = \alpha(f(x))$ are both in A_i which is a contradiction.

□

Theorem 14 (Borsuk's Antipodal Theorem)**.** *An antipode-preserving map $f : \mathbf{S}^n \to \mathbf{S}^n$ is not nullhomotopic.*

Proof. Suppose, indirectly, some antipode-preserving $g : \mathbf{S}^n \to \mathbf{S}^n$ is nullhomotopic. Then g would be extendable to a $G : \mathbf{K}^{n+1} \to \mathbf{S}^n$. Now, we define a homeomorphism from $H : \mathbf{K}^{n+1} \to \mathbf{S}^{n+1}_+$ by, for any $x = \{x_1, x_2, ..., x_{n+1}, 0, 0, ...\} \in \mathbf{K}^{n+1}$,

$$H(x) = \{x_1, x_2, ..., x_{n+1}, 1 - ||x||, 0, 0, ...\}.$$

Clearly, since $||H(x)|| = 1$ and $1 - ||x|| \geq 0$, we have that $H(x) \in \mathbf{S}^{n+1}_+$. From this, we can then define $\phi : \mathbf{S}^{n+1} \to \mathbf{S}^n$ by

$$\phi(y) = \begin{cases} G(H^{-1}(y)) & y \in \mathbf{S}^{n+1}_+ \\ \alpha(G(H^{-1}(\alpha(y)))) & y \in \mathbf{S}^{n+1}_-. \end{cases}$$

Now, for any $y \in \mathbf{S}^{n+1}_+ \bigcap \mathbf{S}^{n+1}_- = \mathbf{S}^n$, we see

$$G(H^{-1}(y)) = G(y) = g(y) = \alpha(g(\alpha(y))) = \alpha(G(\alpha(y))) = \alpha(G(H^{-1}(\alpha(y)))),$$

because g is antipode-preserving, H^{-1} is the identity map on $\mathbf{S}^n$ and $G(y) = g(y)$ on $\mathbf{S}^n$. So ϕ is consistently defined on $\mathbf{S}^n$.

If $y \in \mathbf{S}^{n+1}_+$, then $\alpha(y) \in \mathbf{S}^{n+1}_-$, so

$$\phi(\alpha(y)) = \alpha(G(H^{-1}(\alpha(\alpha(y))))) = \alpha(G(H^{-1}(y))) = \alpha(\phi(y)).$$

This means that we have an antipode-preserving map of $\mathbf{S}^{n+1}$ to $\mathbf{S}^n$, which we cannot have.

□

4.7. Brouwer's Theorem for $\mathbf{K}^n$.

Theorem 15. *The identity map* $id : \boldsymbol{S}^n \to \boldsymbol{S}^n$ *is not nullhomotopic.*

Proof. Since the identity map is antipode-preserving, by Borsuk's Antipodal Theorem, we have our claim. □

Theorem 16 (Bohl's Theorem)**.** *Every continuous* $f : \boldsymbol{K}^n \to E^n$ *satisfies at least one of these two conditions:*

(1) f *has at least one fixed point.*
(2) *There exists some* $y \in \partial\boldsymbol{K}^n$ *such for some* $\lambda \in (0,1)$*,* $y = \lambda \cdot f(y)$*.*

Proof. Suppose, indirectly, that for a continuous function $f : \mathbf{K}^n \to \mathbf{E}^n$, $f(x) \neq x$ for all $x \in \mathbf{K}^n$, and $y \neq \lambda f(y)$ for all $0 < \lambda < 1$ and $y \in \partial\mathbf{K}^n$. Clearly, since $y \in \partial\mathbf{K}^n$, and thus $f(y) \neq y$, we also have $y \neq tf(y)$ when $t = 0$ and $t = 1$.

Let $r : E^n \setminus \{0\} \to \mathbf{S}^{n-1}$ be the map $x \mapsto \frac{x}{||x||}$. Let us define $H : \mathbf{S}^{n-1} \times \mathbf{I} \to \mathbf{S}^{n-1}$ by

$$H(y,t) = \begin{cases} r(y - 2tf[y]), & 0 \leq t \leq \frac{1}{2} \\ r[(2-2t)y - f\{(2-2t)y\}], & \frac{1}{2} \leq t \leq 1. \end{cases}$$

Since $H(y,0) = r(y) = \frac{y}{||y||} = y$ for all $y \in \mathbf{S}^{n-1}$, we can see that when $t = 0$, H is the identity mapping of $\mathbf{S}^{n-1}$. We also have that $H(\mathbf{S}^{n-1}, 1) = r(-f(0))$, because $H(y,1) = r(-f(0))$ for all $y \in \mathbf{S}^{n-1}$ and we know $f(0) \neq 0$. Since H is a continuous function, it would show that the identity map id : $\mathbf{S}^{n-1} \to \mathbf{S}^{n-1}$ is nullhomotopic, which Theorem 14 shows cannot happen. So we have a contradiction.

Thus, we must have that for some $x \in \mathbf{K}^n$, $f(x) = x$, or there exists some $y \in \mathbf{S}^{n-1}$ and $\lambda \in (0,1)$ such that $y = \lambda f(y)$. □

Theorem 17 (Brouwer's Fixed Point Theorem for $\mathbf{K}^n$)**.** *Every continuous* $f : \boldsymbol{K}^n \to \boldsymbol{K}^n$ *has at least one fixed point.*

Proof. We know from Bohl's Theorem that we must have some $x \in \mathbf{K}^n$, $F(x) = x$, or there exists some $y \in \mathbf{S}^{n-1}$ and $\lambda \in (0,1)$ such that $y = \lambda f(y)$. However, since f maps $\mathbf{K}^n$ to itself, we know that $f(\mathbf{S}^{n-1}) \subseteq \mathbf{K}^n$, meaning that for every $y \in \mathbf{S}^{n-1}$ and $\lambda \in (0,1)$, $||y|| > ||\lambda f(y)||$, so $y \neq \lambda f(y)$. Thus we have that there exists some $x \in \mathbf{K}^n$ such that $F(x) = x$. □

Now, we have Brouwer's Fixed Point Theorem for any unit n-ball, for $n \in \mathbb{N}$, which allows us to generalize. For any n-dimensional, compact, convex set H, if we can construct a **homeomorphism** (a continuous bijection with a continuous inverse), $g : H \to \mathbf{K}^n$, then we find that Brouwer's Theorem holds for such a set too. We will not prove this in this paper, but the basic idea is, if such a homeomorphism exists, then for any continuous function $f : H \to H$, there exists a continuous function $h : \mathbf{K}^n \to \mathbf{K}^n$ such that $h = g^{-1}fg$. We know that for some $x \in \mathbf{K}^n$, $h(x) = x$, and from this, we can conclude that $f(g(x)) = g(x)$, meaning that f has a fixed point in H.

5. Permutations on Finite Sets

Our last three theorems were on sets that were infinite in size, or had the potential to be. So, now we look at finite sets. But it is easy to determine if a function on a finite set has fixed points. So, instead, we look at how many permutations have no fixed points.

This section draws heavily from [5].

5.1. **Combinatorics proof.** For any $n \in \mathbb{N}$, we define $[n]$ to be the set $\{1, 2, 3, ..., n\}$. For any $n \in \mathbb{N}$, let $\mathfrak{S}(n)$ be the set of all permutations on $[n]$. Now, for any $R \subseteq [n] \times [n]$, and $\pi \in \mathfrak{S}(n)$, let $f(R, \pi)$ be the number of elements $i \in [n]$ such that $(i, \pi(i)) \in R$. We note that this means that $0 \leq f(R, \pi) \leq \min\{n, |R|\}$. We denote the power set of $[n] \times [n]$ with $P([n] \times [n])$.

Let us define a function $a_n : P([n] \times [n]) \times \mathbb{R} \to \mathbb{R}$ such that

$$a_n(R, t) = \sum_{\pi \in \mathfrak{S}(n)} t^{f(R,\pi)}.$$

Since the highest value $f(R, \pi)$ can take is n, we define coefficients for each $k \in \{0, 1, ..., n\}$ such that

$$a_n(R, t) = \sum_{\pi \in \mathfrak{S}(n)} t^{f(R,\pi)} = \sum_{k=0}^{n} a_{R,k} t^k.$$

We note here that $a_{R,k}$ is the number of permutations, π, on $[n]$ such that there are exactly k elements in $[n]$ such that $(i, \pi(i)) \in R$. For example, if $R = [n] \times [n]$, $a_{R,k} = 0$ for all $k < n$, and $a_{R,n} = |\mathfrak{S}(n)|$.

Now, for each $k \in \mathbb{N}$ and $R \subseteq [n] \times [n]$, let $b_{R,k}$ be the number of k-subsets of R such that there if (i, j) is an element of the subset, there are no elements of the form (i, k) or (k, j) in the subset, for $i, j, k \in [n]$. Another way to put is is that $b_{R,k}$ is the number of k-subsets of R for which no two elements share either of their coordinates. So if $R = \{(1, 1), (1, 2), (3, 2), (4, 4)\}$, then $b_{R,0} = 1$, $b_{R,1} = 4$, $b_{R,2} = 3$, and $b_{R,k} = 0$ for $k > 2$. We define

$$b_n(R, t) = \sum_{k=0}^{n} b_{R,k}(n - k)! t^k.$$

Theorem 18 (Weighted Counting Theorem)**.** *Let $R \subseteq [n] \times [n]$. Then for all $t \in \mathbb{R}$,*

$$a_n(R, t + 1) = b_n(R, t).$$

Proof. For any $\pi \in \mathfrak{S}(n)$, let $G(\pi) = \{(i, \pi(i) \mid i \in [n]\}$. If $Q \subseteq G(\pi) \cap R$, then we weigh it $t^{|Q|}$. So now we will consider two different ways of finding the value of

$$\sum_{\pi \in \mathfrak{S}} \sum_{Q \subseteq G(\pi) \cap R} t^{|Q|}.$$

For each $\pi \in \mathfrak{S}(n)$, let $\kappa_\pi = |G(\pi) \cap R|$. For any $x \in \mathbb{N}_0$, the number of size x subsets of $G(\pi) \cap R$ is $\binom{\kappa_\pi}{x}$. Thus, for a given $\pi \in \mathfrak{S}(n)$, we have that

$$\sum_{Q \subseteq G(\pi) \cap R} t^{|Q|} = \sum_{x=0}^{\kappa_\pi} \binom{\kappa_\pi}{x} t^x.$$

By the Binomial Theorem, we know that

$$\sum_{x=0}^{\kappa_\pi} \binom{\kappa_\pi}{x} t^x = (t + 1)^{\kappa_\pi}.$$

Thus we have

$$\sum_{\pi \in \mathfrak{S}(n)} \sum_{Q \subset G(\pi) \cap R} t^{|Q|} = \sum_{\pi \in \mathfrak{S}(n)} (t + 1)^{|G(\pi) \cap R|}.$$

Clearly $f(R, \pi) = |G(\pi) \cap R|$, so we have

$$\sum_{\pi \in \mathfrak{S}(n)} \sum_{Q \subset G(\pi) \cap R} t^{|Q|} = \sum_{\pi \in \mathfrak{S}(n)} (t + 1)^{f(R,\pi)} = a_n(R, t + 1).$$

Now, we observe that in our sum, for any $Q \subseteq R$, we add the weight $t^{|Q|}$ the number of times that there is a permutation π such that $Q \subseteq G(\pi)$. So,

$$\sum_{\pi \in \mathfrak{S}(n)} \sum_{Q \subset G(\pi) \cap R} t^{|Q|} = \sum_{Q \in R} |\{\pi \in \mathfrak{S}(n) \mid G(\pi) \supseteq Q\}| t^{|Q|}.$$

Since, for every $\pi \in \mathfrak{S}(n)$, every element of $G(\pi)$ has a first coordinate distinct from all of the other elements, and π is a permutation, meaning that it sends no two elements of $[n]$ to the same value, Q can be a subset of $G(\pi)$, for some $\pi \in \mathfrak{S}(\mathfrak{n})$, if and only if no two of its elements share either of their coordinates. Thus, for each $k \in \{0, 1, ..., n\}$, we have that $b_{R,k}$ is the number of k sized subsets of R such that there exists some permutation $\pi \in \mathfrak{S}(n)$ such that that subset of R is also a subset of $G(\pi)$. In addition, the number of permutations of $[n]$ with $|Q|$ elements mapping to set values is $(n - |Q|)!$. Thus we have

$$\sum_{\pi \in \mathfrak{S}(n)} \sum_{Q \subset G(\pi) \cap R} t^{|Q|} = \sum_{Q \in R} |\{\pi \in \mathfrak{S}(n) \mid G(\pi) \supseteq Q\}| t^{|Q|} = \sum_{k=0}^{n} b_{R,k}(n-k)! t^k = b_n(R, t).$$

And thus we have our claim.

□

Corollary 1. *The number of permutations of* $[n]$ *with exactly* k *fixed points is*

$$\sum_{p=0}^{n-k} (-1)^p \frac{n!}{p!k!}.$$

Proof. Suppose that $R = \{(1,1), (2,2), ..., (n,n)\}$. Then for each $k \in \{0, 1, ..., n\}$, $a_{R,k}$ is the number of permutations on $[n]$ with exactly k fixed points. Now, since $a_n(R, t+1) = b_n(R, t)$ for all $\mathbb{R}$, and both are polynomials in t, we know that they are both infinitely differentiable in $\mathbb{R}$, and that their derivatives must be equal. So we have that

$$\frac{d^i}{dt^i} a_n(R, t+1) = \sum_{p=0}^{n-i} \frac{(p+i)!}{p!} a_{R,p+i}(t+1)^p = \sum_{p=0}^{n-i} \frac{(p+i)!}{p!} b_{R,p+i}(n-p-i)! t^p = \frac{d^i}{dt^i} b_n(R, t).$$

Thus, when we take the i^{th} derivative and $t = -1$, we get

$$\frac{d^i}{dt^i} a_n(R, 0) = i! \cdot a_{R,i} = \sum_{p=0}^{n-i} \frac{(p+i)!(n-p-i)!}{p!} b_{R,p+i}(-1)^p = \frac{d^i}{dt^i} b_n(R, -1).$$

Now, since due to how we defined R, $b_{R,k}$ ends up being the number of k-sized subsets of R. Thus, this finally gives us that the number of permutations with k fixed points on $[n]$ is

$$a_{R,k} = \sum_{p=0}^{n-k} (-1)^p \frac{(p+k)!(n-p-k)!}{p!k!} \binom{n}{k+p}$$

which yields

$$a_{R,k} = \sum_{p=0}^{n-k} (-1)^p \frac{n!}{p!k!}.$$

□

We can see from this, the number of **derangements**, permutations with no fixed points, of $[n]$ is

$$a_{R,0} = \sum_{p=0}^{n} (-1)^p \frac{n!}{p!}.$$

5.2. **Set Method.** This first method we just used is nice in the sense that it allows us to find the number of permutations of $[n]$ with **any** number of fixed points. However, if we are just looking for the total number of derangements of $[n]$ and nothing else, then there is an easier proof to follow. To make the proof, need the Inclusion-Exclusion Principle. We will not provide the proof here, though one could find it with some reading in set theory.

Theorem 19 (Inclusion-Exclusion Principle)**.** *Let $n \in \mathbb{N}$ and suppose that A_1, A_2, ..., A_n are all finite sets. Then*

$$\left|\bigcup_{i=1}^{n} A_i\right| = \sum_{k=1}^{n} (-1)^{k-1} \cdot \sum_{1 \le i_1 < \ldots < i_k \le n} \left|\bigcap_{j \in \{i_1, \ldots, i_k\}} A_j\right|.$$

The idea of Theorem 19 is that if you count all of the elements in each of your sets, then any element that two or more sets share will would be counted more than once. So, for every possible pairing of sets, you count the elements in their intersection, and subtract it from your original total. However, the elements that were shared by three or more sets were again overcounted, and thus must be added back into the set by counting the elements in each intersection of three sets and adding the total back in. But again, some elements may have been overcounted. We keep correcting like this until we have gone through all the possible intersections of sets, and thus find our total.

So, if A_1, A_2, and A_3 are all finite sets,

$$|A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2|,$$

and

$$|A_1 \cup A_2 \cup A_3| = |A_1| + |A_2| + |A_3| - |A_1 \cap A_2| - |A_1 \cap A_3| - |A_2 \cap A_3| + |A_1 \cap A_2 \cap A_3|.$$

Now, we can prove the number of derangements of a finite set.

Theorem 20 (Derangements of a Finite Set)**.** *Let $n \in \mathbb{N}$. The total number of derangements of $[n]$ is*

$$\sum_{i=0}^{n} (-1)^i \frac{n!}{i!}.$$

Proof. For each $i \in [n]$, let A_i be the set of permutations of $[n]$ such that i is a fixed point. We note that for any elements $i_1, i_2 \in [n]$, $|A_{i_1}| = |A_{i_2}|$. In fact for any sets $I = \{i_1, i_2, ..., i_k\} \subseteq [n]$ and $J = \{j_1, j_2, ..., j_k\} \subseteq [n]$, we have

$$\left|\bigcap_{i \in I} A_i\right| = \left|\bigcap_{j \in J} A_j\right|.$$

In other words, the number of permutations of $[n]$ with a specific set of k elements fixed is the same as the number of permutations with a different set of k elements fixed.

Clearly, the set of all permutations of $[n]$ with at least one fixed point is

$$B = \bigcup_{i=1}^{n} A_i.$$

So then, the set of all derangements of $[n]$ is $D = \mathfrak{S}(n) \setminus B$, which means that we have that the number of derangements of $[n]$ is

$$|D| = |\mathfrak{S}(n)| - |B|.$$

To find the size of B, we can use Theorem 19, since it is the union of a finite number of sets. Thus we have

$$|B| = \sum_{1 \leq i_1 \leq n} |A_i| - \sum_{\substack{1 \leq i_1 < n \\ i_1 < i_2 \leq n}} |A_{i_1} \cap A_{i_2}| + \sum_{\substack{1 \leq i_1 < n-1 \\ i_1 < i_2 < n \\ i_2 < i_3 \leq n}} |A_{i_1} \cap A_{i_2} \cap A_{i_3}| - ... + (-1)^{n-1} \left| \bigcap_{j=1}^{n} A_{i_j} \right|$$

. Now, the number of permutations where $i_1, i_2, ..., i_k \in [n]$ are fixed points is $(n-k)!$. The number of ways that we can choose k elements of $[n]$ to be fixed points is $\binom{n}{k}$. Thus our sum becomes

$$\begin{aligned}
|B| &= \binom{n}{1}(n-1)! - \binom{n}{2}(n-2)! + ... + (-1)^{n-1}\binom{n}{n}(n-n)! \\
&= \frac{n!}{(n-1)!1!}(n-1)! - \frac{n!}{(n-2)!2!}(n-2)! + ... + (-1)^{n-1}\frac{n!}{(n-n)!n!}(n-n)! \\
&= \frac{n!}{1!} - \frac{n!}{2!} + ... + (-1)^{n-1}\frac{n!}{n!} \\
&= \sum_{i=1}^{n} (-1)^{i-1}\frac{n!}{i!}.
\end{aligned}$$

The number of total possible permutations of $[n]$ is $n!$, so we have that the total number of derangements is

$$\begin{aligned}
|D| &= |\mathfrak{S}| - |B| \\
&= n! - \sum_{i=1}^{n} (-1)^{i-1}\frac{n!}{i!} \\
&= \frac{n!}{0!} + \sum_{i=1}^{n} (-1)^{i}\frac{n!}{i!} \\
&= \sum_{i=0}^{n} (-1)^{i}\frac{n!}{i!}.
\end{aligned}$$

And thus we have our claim.

□

6. Conclusion

So we do indeed find that Fixed Point Theory reaches out and involves many different fields of mathematics and systems. Each of the theorems had its own unique conditions and results. These four only covered a small amount of all of Fixed Point Theory. For more, see [4].

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