COMPLEX ANALYSIS EXAMPLES

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1. Euler's Identity

(1) (S 2015 # 1) Describe all the values of $(-1)^i$, where $i = \sqrt{-1}$.

Proof. We have, for any $n \in \mathbb{Z}$

$$(-1)^i = (e^{i\pi + 2n\pi i})^i = e^{-(2n+1)\pi}.$$

□

(2) (F 2014 # 1) Describe all the values of i^i, where $i = \sqrt{-1}$.

Proof. We have, for any $n \in \mathbb{Z}$

$$i^i = (e^{i\frac{\pi}{2} + 2n\pi i})^i = e^{-(2n+\frac{1}{2})\pi}.$$

□

2. Convergence of Series

Theorem 1 (Dirichlet's Test)**.** *If $(a_n)_{n\in\mathbb{N}}$ is a sequence of positive real numbers and $(b_n)_{n\in\mathbb{N}}$ is a sequence of complex numbers such that*

- $a_{n+1} \leq a$
- $\lim_{n\to\infty} a_n = 0$
- $\left|\sum_{n=1}^{N} b_n\right| < C$ *for some constant C, for all $N \in \mathbb{N}$,*

then $\sum_{n=1}^{\infty} a_n b_n$ converges.

(1) (F 2016 # 2) Show that $\sum_n \frac{z^n}{n}$ converges at all points of the unit circle $|z| = 1$ except $z = 1$.

Proof. Let $|z| = 1$ with $z \neq 1$. For all $N \in \mathbb{N}$,

$$\left|\sum_{n=1}^{N} z^n\right| = \left|\frac{z^{n+1} - 1}{z - 1}\right| \leq \frac{2}{|z-1|}.$$

$(\frac{1}{n})_{n\in\mathbb{N}}$ is a decreasing sequence of real numbers that converges to 0, So, by Dirichlet's Test for convergence, we know that

$$\sum_{n=1}^{\infty} \frac{z^n}{n}$$

converges.

□

3. Laurent Expansions

Theorem 2 (Laurent expansions on annuli)**.** *A holomorphic function f in the annulus $\Omega = \{z \in \mathbb{C} : r < |z-z_0| < R\}$ has the Laurent expansion*

$$f(z) = \sum_{n\in\mathbb{Z}} c_n (z-z_0)^n$$

absolutely convergent in Ω, and uniformly convergent on compact subsets.

The Laurent coefficients c_n are given by the Cauchy Formulas for Laurent Coefficients

$$c_n = \begin{cases} \frac{1}{2\pi i}\int_{\gamma_R} \frac{f(w)}{(w-z_0)^{n+1}} & n \geq 0 \\ \frac{1}{2\pi i}\int_{\gamma_r} \frac{f(w)}{(w-z_0)^{-n+1}} & n < 0, \end{cases}$$

where γ_R is any circle about z_0 of radius slightly less than R, and γ_r is a circle about z_0 of radius slightly more than r. These coefficients are unique for $r < |z-z_0| < R$.

(1) (S 2018 # 1) Write three terms of the Laurent expansion of $f(z) = \frac{1}{z(z-1)(z-2)}$ centered at 0 and convergent in $1 < |z| < 2$.

Proof. Partial fraction decomposition gives us that

$$f(z) = \frac{1}{2z} + \frac{-1}{z-1} + \frac{1}{2(z-2)}.$$

$\frac{1}{2z}$, $\frac{-1}{z-1}$, and $\frac{1}{2(z-2)}$ are holomorphic on $1 < |z| < 2$, so they have Laurent series centered at 0. We have that

$$\frac{1}{z-1} = \frac{1}{z}\frac{1}{1-\frac{1}{z}} = \frac{1}{z}\sum_{n=0}^{\infty}\frac{1}{z^n} = \sum_{n=1}^{\infty}\frac{1}{z^n},$$

and

$$\frac{1}{z-2} = \frac{-1}{2}\frac{1}{1-\frac{z}{2}} = \frac{-1}{2}\sum_{n=0}^{\infty}\frac{z^n}{2^n}.$$

Thus

$$f(z) = \frac{1}{2}\frac{1}{z} - \sum_{n=1}^{\infty}\frac{1}{z^n} - \frac{1}{4}\sum_{n=0}^{\infty}\frac{z^n}{2^n} = -\sum_{n=2}^{\infty}\frac{1}{z^n} - \frac{1}{4}\sum_{n=-1}^{\infty}\frac{z^n}{2^n}.$$

Pick any three terms. □

(2) (F 2017 # 1) Write three terms of the Laurent expansion of $f(z) = \frac{e^z}{z-1}$ centered at 0 and convergent in $1 < |z|$.

Proof. We know that e^z is holomorphic everywhere, and is given by the power series

$$e^z = \sum_{n=0}^{\infty}\frac{z^n}{n!}.$$

We also know $\frac{1}{z-1}$ is holomorphic on the annulus $1 < |z|$, with Laurent series

$$\frac{1}{z-1} = \frac{1}{z}\sum_{n=0}^{\infty}\frac{1}{z^n} = \sum_{n=1}^{\infty}\frac{1}{z^n}.$$

Thus, we have that

$$f(z) = \left(\sum_{n=0}^{\infty}\frac{z^n}{n!}\right)\left(\sum_{m=1}^{\infty}\frac{1}{z^m}\right).$$

So, we have that for any $k \in \mathbb{N}_0$,

$$c_k = \sum_{j=k+1}^{\infty} \frac{1}{j!}$$

and

$$c_{-k} = \sum_{j=0}^{\infty} \frac{1}{j!} = e - 1.$$

□

(3) (S 2017 # 1) Write three terms of the Laurent expansion of $f(z) = \frac{e^z-1}{z(z-1)}$ centered at 0 and convergent in $1 < |z|$.

Proof. We know that $\frac{e^z-1}{z}$ has a removable pole at 0, and so is holomorphic everywhere, and is given by the power series

$$\frac{e^z - 1}{z} = \sum_{n=1}^{\infty} \frac{z^{n-1}}{n!}.$$

We also know $\frac{1}{z-1}$ is holomorphic on the annulus $1 < |z|$, with Laurent series

$$\frac{1}{z-1} = \frac{1}{z} \sum_{n=0}^{\infty} \frac{1}{z^n} = \sum_{n=1}^{\infty} \frac{1}{z^n}.$$

Thus, we have that

$$f(z) = \left(\sum_{n=1}^{\infty} \frac{z^{n-1}}{n!} \right) \left(\sum_{m=1}^{\infty} \frac{1}{z^m} \right).$$

So, we have that for any $k \in \mathbb{N}_0$,

$$c_k = \sum_{j=k+2}^{\infty} \frac{1}{j!}$$

and for $k > 0$,

$$c_{-k} = \sum_{j=1}^{\infty} \frac{1}{j!} = e - 1.$$

□

(4) (F 2016 # 1) Write three terms of the Laurent expansion of $f(z) = \frac{1}{z(z-1)(z+1)}$ in the annulus $1 < |z|$.

Proof. Partial fraction decomposition gives us that

$$f(z) = \frac{-1}{z} + \frac{1}{2(z-1)} + \frac{1}{2(z+1)}.$$

$\frac{-1}{z}$, $\frac{1}{2(z-1)}$, and $\frac{1}{2(z+1)}$ are holomorphic on $1 < |z|$, so they have Laurent series centered at 0. We have that

$$\frac{1}{z-1} = \frac{1}{z} \frac{1}{1 - \frac{1}{z}} = \frac{1}{z} \sum_{n=0}^{\infty} \frac{1}{z^n} = \sum_{n=1}^{\infty} \frac{1}{z^n},$$

and

$$\frac{1}{z+1} = \frac{1}{z} \frac{1}{1-(-z)} = \frac{1}{z} \sum_{n=0}^{\infty} \frac{1}{(-z)^n} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{z^n}.$$

Thus

$$f(z) = \frac{-1}{z} + \frac{1}{2} \sum_{n=1}^{\infty} \frac{1}{z^n} + \frac{1}{2} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{z^n} = \frac{-1}{z} + \sum_{n=1}^{\infty} \frac{1}{z^{2n+1}}.$$

Pick any three terms. □

(5) (S 2016 # 1) Write the first three non-zero terms of the Laurent expansion of $f(z) = \frac{1}{z^2-1}$ in the annulus $1 < |z|$.

Proof. Partial fraction decomposition gives us that

$$f(z) = \frac{1}{2(z-1)} - \frac{1}{2(z+1)}.$$

$\frac{1}{2(z-1)}$ and $\frac{-1}{2(z+1)}$ are holomorphic on $1 < |z|$, so they have Laurent series centered at 0. We have that

$$\frac{1}{z-1} = \frac{1}{z}\frac{1}{1-\frac{1}{z}} = \frac{1}{z}\sum_{n=0}^{\infty}\frac{1}{z^n} = \sum_{n=1}^{\infty}\frac{1}{z^n},$$

and

$$\frac{1}{z+1} = \frac{1}{z}\frac{1}{1-\frac{-1}{z}} = \frac{1}{z}\sum_{n=0}^{\infty}\frac{1}{(-z)^n} = \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{z^n},$$

Thus

$$f(z) = \frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{z^n} - \frac{1}{2}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{z^n} = \sum_{n=1}^{\infty}\frac{1}{z^{2n}}.$$

Pick the first three terms. □

(6) (F 2015 # 1) Write three terms of the Laurent expansion of $f(z) = \frac{1}{z(z-1)(z+1)}$ in the annulus $0 < |z| < 1$.

Proof. Partial fraction decomposition gives us that

$$f(z) = \frac{-1}{z} + \frac{1}{2(z-1)} + \frac{1}{2(z+1)}.$$

$\frac{-1}{z}$, $\frac{1}{2(z-1)}$, and $\frac{1}{2(z+1)}$ are holomorphic on $0 < |z| < 1$, so they have Laurent series centered at 0. We have that

$$\frac{1}{z-1} = \frac{-1}{1-z} = -\sum_{n=0}^{\infty} z^n,$$

and

$$\frac{1}{z+1} = \frac{1}{1-(-z)} = \sum_{n=0}^{\infty}(-z)^n.$$

Thus

$$f(z) = \frac{-1}{z} - \frac{1}{2}\sum_{n=0}^{\infty} z^n + \frac{1}{2}\sum_{n=0}^{\infty}(-z)^n = \frac{-1}{z} - \sum_{n=0}^{\infty} z^{2n+1}.$$

Pick any three terms. □

(7) (S 2015 # 2) Write three terms of the Laurent expansion of $f(z) = \frac{1}{z(z-1)(z-2)}$ in the annulus $1 < |z| < 2$.

Proof. See above. □

(8) (F 2014 # 4) Write four terms of the Laurent expansion of $f(z) = \frac{1}{z(z-1)}$ in the annulus $1 < |z|$.

Proof. Partial fraction decomposition gives us that

$$f(z) = \frac{-1}{z} + \frac{1}{z-1}.$$

Both $\frac{-1}{z}$ and $\frac{1}{z-1}$ are holomorphic on $1 < |z|$, so they both have Laurent series centered at 0. We have that

$$\frac{1}{z-1} = \frac{1}{z}\frac{1}{1-\frac{1}{z}} = \frac{1}{z}\sum_{n=0}^{\infty}\frac{1}{z^n} = \sum_{n=1}^{\infty}\frac{1}{z^n}.$$

Thus

$$f(z) = \frac{-1}{z} + \sum_{n=1}^{\infty} \frac{1}{z^n} = \sum_{n=2}^{\infty} \frac{1}{z^n}.$$

□

3.1. **Other examples.**

(1) Determine the Laurent series expansions of $f(z) = \frac{1}{(z-1)^4}$ in the annulus $1 < |z|$ and the annulus $|z-1| > 0$.

Proof. We know $\frac{1}{z-1}$ is holomorphic in $1 < |z|$, with Laurent series expansion

$$\frac{1}{z-1} = \frac{1}{z}\frac{1}{1-\frac{1}{z}} = \frac{1}{z}\sum_{n=0}^{\infty}\frac{1}{z^n} = \sum_{n=1}^{\infty}\frac{1}{z^n}.$$

Differentiating termwise three times, we have

$$\frac{-6}{(z-1)^4} = \sum_{n=1}^{\infty} \frac{-n(n+1)(n+2)}{z^{n+3}},$$

so

$$f(z) = \frac{1}{6}\sum_{n=1}^{\infty} \frac{n(n+1)(n+2)}{z^{n+3}}.$$

On the annulus $|z-1| > 0$, $\frac{1}{(z-1)^4}$ is the Laurent series expansion.

□

4. Evaluation of Integrals

Theorem 3 (Residues)**.** *Let γ be a simple closed curve in a convex open Ω, and f holomorphic on $\Omega \setminus \{z_1, ..., z_n\}$. Then, if $z_1, ..., z_n$ are inside γ,*

$$\int_\gamma f = 2\pi i \cdot \sum_{j=1}^{n} Res_{z=z_j} f,$$

where

$$Res_{z=z_j} f = c_{-1},$$

from the Laurent expansion of f centered at z_j.

(1) (S 2018 # 3) Evaluate $\int_0^\infty \frac{\cos(tx)dx}{1+x^2}$ for $t \in \mathbb{R}$.

Proof. We know that for all $t \in \mathbb{R}$,

$$\int_{-\infty}^{0} \frac{\cos(tx)}{1+x^2}dx = \int_{\infty}^{0} \frac{\cos(t(-x))}{1+(-x)^2}d(-x) = \int_0^\infty \frac{\cos(tx)}{1+x^2}dx.$$

Suppose that $t \geq 0$. Let $f(z) = \frac{e^{itz}}{1+z^2}$ for $z \in \mathbb{C}$ and γ_R be the simple closed path from $-R$ to R along the real line, then counterclockwise along the arc, a_R, of the semicircle of radius R in the upper half plane.

We see that f has poles at exactly the primitive fourth roots of unity, only one of which are in the upper half plane. The Residue Theorem gives us that for all $R > 1$,

$$\int_{\gamma_R} f(z)dz = 2\pi i \cdot \mathrm{Res}_{z=i} f = 2\pi i \cdot \frac{e^{-t}}{2i} = \frac{\pi}{e^t}.$$

We then have

$$\frac{\pi}{e^t} = \int_{\gamma_R} f(z)dz = \int_{-R}^{R} f(z)dz + \int_{a_R} f(z)dz.$$

We have that
$$\left|\int_{a_R} f(z)dz\right| \leq \text{length}(a_R)\cdot \sup_{z\in a_R}\left|\frac{e^{itz}}{1+z^2}\right| \leq \pi R\cdot\frac{1}{R^2-1}\cdot\sup_{0<x\leq R} e^{-tx} = \pi R\cdot\frac{1}{R^2-1},$$
which goes to zero at R goes to ∞.

Thus, we have that
$$\begin{aligned}\frac{\pi}{e^t} &= \lim_{R\to\infty}\int_{\gamma_R} f(z)dz\\ &= \lim_{R\to\infty}\left(\int_{-R}^{R} f(z)dz + \int_{a_R} f(z)dz\right)\\ &= \int_{-\infty}^{\infty} f(z)dz.\end{aligned}$$

For $t<0$, we see that
$$\int_{-\infty}^{\infty}\frac{e^{itx}dx}{1+x^2} = \int_{\infty}^{-\infty}\frac{e^{-i|t|(-x)}d(-x)}{1+(-x)^2} = \int_{-\infty}^{\infty}\frac{e^{i|t|x}dx}{1+x^2} = \frac{\pi}{e^{|t|}}.$$

This gives us, for all $t\in\mathbb{R}$,
$$\begin{aligned}2\int_0^{\infty}\frac{\cos(tx)}{1+x^2}dx &= \int_{-\infty}^{\infty}\frac{\cos(tx)}{1+x^2}dx\\ &= \int_{-\infty}^{\infty}\frac{e^{itx}+e^{i(-t)x}}{2+2x^2}dx\\ &= \frac{\pi}{e^{|t|}}.\end{aligned}$$

Thus
$$\int_0^{\infty}\frac{\cos(tx)}{1+x^2}dx = \frac{\pi}{2e^{|t|}}.$$

□

(2) (F 2017 # 5) Evaluate $\int_0^\infty \frac{x^{1/3}dx}{1+x^2}$.

Proof. We have that
$$\int_{-\infty}^{0}\frac{x^{1/3}dx}{1+x^2} = -\int_0^{\infty}\frac{(-x)^{1/3}d(-x)}{1+(-x)^2} = e^{\pi i/3}\int_0^{\infty}\frac{x^{1/3}dx}{1+x^2}.$$

Let $f(z)=\frac{z^{1/3}}{1+z^2}$ for $z\in\mathbb{C}$ and γ_R be the simple closed path from $-R$ to R along the real line, then counterclockwise along the arc, a_R, of the semicircle of radius R in the upper half plane.

We see that f has poles at exactly the primitive fourth roots of unity, only one of which is in the upper half plan. The Residue Theorem gives us that for all $R>1$,
$$\int_{\gamma_R} f(z)dz = 2\pi i\text{Res}_{z=i}f = 2\pi i\cdot\frac{i^{1/3}}{2i} = \pi e^{\pi i/6}.$$

We then have
$$\pi e^{\pi i/6} = \int_{\gamma_R} f(z)dz = \int_{-R}^{R} f(z)dz + \int_{a_R} f(z)dz.$$

We have that
$$\left|\int_{a_R} f(z)dz\right| \leq \text{length}(a_R)\cdot\sup_{z\in a_R}\left|\frac{z^{1/3}}{1+z^2}\right| \leq \pi R\cdot\frac{R^{1/3}}{R^2-1},$$
which goes to zero at R goes to ∞.

Thus, we have that

$$\begin{aligned}\pi e^{\pi i/6} &= \lim_{R\to\infty}\int_{\gamma_R} f(z)dz \\ &= \lim_{R\to\infty}\left(\int_{-R}^{R} f(z)dz + \int_{a_R} f(z)dz\right) \\ &= \int_{-\infty}^{\infty} f(z)dz \\ &= (e^{\pi i/3}+1)\int_0^{\infty} f(z)dz,\end{aligned}$$

so

$$\int_0^{\infty} f(z)dz = \frac{\pi e^{\pi i/6}}{e^{\pi i/3}+1} = \frac{\pi}{\sqrt{3}}.$$

□

(3) (S 2017 # 3) Evaluate $\int_{-\infty}^{\infty} \frac{e^{ix}dx}{1+x^2}$.

Proof. Let $f(z) = \frac{e^{iz}}{1+z^2}$ for $z \in \mathbb{C}$ and γ_R be the simple closed path from $-R$ to R along the real line, then counterclockwise along the arc, a_R, of the semicircle of radius R in the upper half plane.

We see that f has poles at exactly the primitive fourth roots of unity, only one of which is in the upper half plan. The Residue Theorem gives us that for all $R > 1$,

$$\int_{\gamma_R} f(z)dz = 2\pi i \mathrm{Res}_{z=i} f = 2\pi i \cdot \frac{e^{-1}}{2i} = \frac{\pi}{e}.$$

We then have

$$\frac{\pi}{e} = \int_{\gamma_R} f(z)dz = \int_{-R}^{R} f(z)dz + \int_{a_R} f(z)dz.$$

We have that

$$\left|\int_{a_R} f(z)dz\right| \le \mathrm{length}(a_R)\cdot \sup_{z\in a_R}\left|\frac{e^{iz}}{1+z^2}\right| \le \pi R \cdot \frac{1}{R^2-1},$$

which goes to zero at R goes to ∞.

Thus, we have that

$$\begin{aligned}\frac{\pi}{e} &= \lim_{R\to\infty}\int_{\gamma_R} f(z)dz \\ &= \lim_{R\to\infty}\left(\int_{-R}^{R} f(z)dz + \int_{a_R} f(z)dz\right) \\ &= \int_{-\infty}^{\infty} f(z)dz.\end{aligned}$$

□

(4) (F 2016 # 5) Evaluate $\int_0^{\infty} \frac{\sqrt[3]{x}dx}{1+x^2}$.

Proof. See above. □

(5) (S 2016 # 3) Evaluate $\int_{-\infty}^{\infty} \frac{dx}{4+5x^2+x^4}$.

Proof. Let $f(z) = \frac{1}{4+5z^2+z^4}$ for $z \in \mathbb{C}$ and γ_R be the simple closed path from $-R$ to R along the real line, then counterclockwise along the arc, a_R, of the semicircle of radius R in the upper half plane.

We see that f has poles at exactly $z = \pm i$ and $z = \pm 2i$, only two of which are in the upper half plan. The Residue Theorem gives us that for all $R > 2$,

$$\int_{\gamma_R} f(z)dz = 2\pi i \left(\mathrm{Res}_{z=i} f + \mathrm{Res}_{z=2i} f\right) = 2\pi i \cdot \left(\frac{1}{6i} + \frac{1}{-12i}\right) = \frac{\pi}{6}.$$

We then have

$$\frac{\pi}{6} = \int_{\gamma_R} f(z)dz = \int_{-R}^{R} f(z)dz + \int_{a_R} f(z)dz.$$

We have that, for sufficiently large R,

$$\left|\int_{a_R} f(z)dz\right| \leq \mathrm{length}(a_R) \cdot \sup_{z \in a_R} \left|\frac{1}{4 + 5z^2 + z^4}\right| \leq \pi R \cdot \frac{1}{R^4 - 5R^2 - 4},$$

which goes to zero at R goes to ∞.

Thus, we have that

$$\begin{aligned} \frac{\pi}{6} &= \lim_{R \to \infty} \int_{\gamma_R} f(z)dz \\ &= \lim_{R \to \infty} \left(\int_{-R}^{R} f(z)dz + \int_{a_R} f(z)dz\right) \\ &= \int_{-\infty}^{\infty} f(z)dz. \end{aligned}$$

□

(6) (F 2015 # 5) Evaluate $\int_0^\infty \frac{\sqrt{x}dx}{1+x^2}$.

Proof. We have that

$$\int_{-\infty}^{0} \frac{x^{1/2}dx}{1 + x^2} = -\int_{0}^{\infty} \frac{(-x)^{1/2}d(-x)}{1 + (-x)^2} = i\int_{0}^{\infty} \frac{x^{1/2}dx}{1 + x^2}.$$

Let $f(z) = \frac{z^{1/2}}{1+z^2}$ for $z \in \mathbb{C}$ and γ_R be the simple closed path from $-R$ to R along the real line, then counterclockwise along the arc, a_R, of the semicircle of radius R in the upper half plane.

We see that f has poles at exactly the primitive fourth roots of unity, only one of which is in the upper half plan. The Residue Theorem gives us that for all $R > 1$,

$$\int_{\gamma_R} f(z)dz = 2\pi i \mathrm{Res}_{z=i} f = 2\pi i \cdot \frac{i^{1/2}}{2i} = \pi e^{\pi i/4}.$$

We then have

$$\pi e^{\pi i/4} = \int_{\gamma_R} f(z)dz = \int_{-R}^{R} f(z)dz + \int_{a_R} f(z)dz.$$

We have that

$$\left|\int_{a_R} f(z)dz\right| \leq \mathrm{length}(a_R) \cdot \sup_{z \in a_R} \left|\frac{z^{1/2}}{1 + z^2}\right| \leq \pi R \cdot \frac{R^{1/2}}{R^2 - 1},$$

which goes to zero at R goes to ∞.

Thus, we have that

$$\begin{aligned}\pi e^{\pi i/4} &= \lim_{R\to\infty}\int_{\gamma_R} f(z)dz \\ &= \lim_{R\to\infty}\left(\int_{-R}^{R} f(z)dz + \int_{a_R} f(z)dz\right) \\ &= \int_{-\infty}^{\infty} f(z)dz \\ &= (i+1)\int_0^{\infty} f(z)dz,\end{aligned}$$

so

$$\int_0^{\infty} f(z)dz = \frac{\pi e^{\pi i/4}}{i+1} = \frac{\pi}{\sqrt{2}}.$$

□

(7) (S 2015 # 5) Evaluate $\int_{-\infty}^{\infty} \frac{e^{i\xi x}dx}{1+x^2}$ for $\xi \in \mathbb{R}$.

Proof. Suppose that $\xi \geq 0$. Let $f(z) = \frac{e^{i\xi z}}{1+z^2}$ for $z \in \mathbb{C}$ and γ_R be the simple closed path from $-R$ to R along the real line, then counterclockwise along the arc, a_R, of the semicircle of radius R in the upper half plane.

We see that f has poles at exactly the primitive fourth roots of unity, only one of which are in the upper half plane. The Residue Theorem gives us that for all $R > 1$,

$$\int_{\gamma_R} f(z)dz = 2\pi i \cdot \mathrm{Res}_{z=i} f = 2\pi i \cdot \frac{e^{-\xi}}{2i} = \frac{\pi}{e^{\xi}}.$$

We then have

$$\frac{\pi}{e^{\xi}} = \int_{\gamma_R} f(z)dz = \int_{-R}^{R} f(z)dz + \int_{a_R} f(z)dz.$$

We have that

$$\left|\int_{a_R} f(z)dz\right| \leq \mathrm{length}(a_R)\cdot \sup_{z\in a_R}\left|\frac{e^{itz}}{1+z^2}\right| \leq \pi R\cdot\frac{1}{R^2-1}\cdot\sup_{0<x\leq R} e^{-tx} = \pi R\cdot\frac{1}{R^2-1},$$

which goes to zero at R goes to ∞.

Thus, we have that

$$\begin{aligned}\frac{\pi}{e^{\xi}} &= \lim_{R\to\infty}\int_{\gamma_R} f(z)dz \\ &= \lim_{R\to\infty}\left(\int_{-R}^{R} f(z)dz + \int_{a_R} f(z)dz\right) \\ &= \int_{-\infty}^{\infty} f(z)dz.\end{aligned}$$

For $\xi < 0$, we see that

$$\int_{-\infty}^{\infty}\frac{e^{i\xi x}dx}{1+x^2} = \int_{\infty}^{-\infty}\frac{e^{-i|\xi|(-x)}d(-x)}{1+(-x)^2} = \int_{-\infty}^{\infty}\frac{e^{i|\xi|x}dx}{1+x^2} = \frac{\pi}{e^{|\xi|}}.$$

□

(8) (F 2015 # 5) Evaluate $\int_{-\infty}^{\infty}\frac{dx}{1+x^4}$.

Proof. Let $f(z) = \frac{1}{1+z^4}$ for $z \in \mathbb{C}$ and γ_R be the simple closed path from $-R$ to R along the real line, then counterclockwise along the arc, a_R, of the semicircle of radius R in the upper half plane.

We see that f has poles at exactly the primitive eighth roots of unity, only two of which ($\omega = e^{\pi i/4}$ and ω^3) are in the upper half plan. The Residue Theorem gives us that for all $R > 1$,

$$\int_{\gamma_R} f(z)dz = 2\pi i \left(\text{Res}_{z=\omega} f + \text{Res}_{z=\omega^3} f\right) = 2\pi i \left(\frac{1}{\omega^3} \frac{1}{(1-\omega^2)(1-\omega^4)(1-\omega^6)} + \frac{1}{\omega^9} \frac{1}{(1-\omega^{-2})(1-\omega^2)(1-\omega^4)} \right)$$

which simplifies to

$$\frac{2\pi i}{4} \left(\frac{1}{\omega^3} + \frac{1}{\omega^9} \right) = \frac{\pi}{\sqrt{2}}.$$

We then have

$$\frac{\pi}{\sqrt{2}} = \int_{\gamma_R} f(z)dz = \int_{-R}^{R} f(z)dz + \int_{a_R} f(z)dz.$$

We have that

$$\left| \int_{a_R} f(z)dz \right| \leq \text{length}(a_R) \cdot \sup_{z \in a_R} \left| \frac{1}{1+z^4} \right| \leq \pi R \cdot \frac{1}{R^4 - 1},$$

which goes to zero at R goes to ∞.

Thus, we have that

$$\begin{aligned} \frac{\pi}{\sqrt{2}} &= \lim_{R\to\infty} \int_{\gamma_R} f(z)dz \\ &= \lim_{R\to\infty} \left(\int_{-R}^{R} f(z)dz + \int_{a_R} f(z)dz \right) \\ &= \int_{-\infty}^{\infty} f(z)dz. \end{aligned}$$

□

5. Zeros of Functions

Theorem 4 (Identity Principle)**.** *Suppose f and g are holomorphic functions on a connected open set Ω, and let $\{z_j\}_{j\in\mathbb{N}}$ with $\lim_{j\to\infty} z_j = z$ for some $z \in \Omega$. Then $f = g$ on Ω.*

Theorem 5 (Rouché's Theorem)**.** *Let f by holomorphic on an open set U containing a simple closed path γ and containing the intereior of γ. Suppose that f does not vanish on γ. If another holomorphic function g on U satisfies*

$$|f(z) - g(z)| < |f(z)| \quad (\textit{for all } z \in \gamma)$$

then the number of zeros of g inside γ is the same as the number of zeros of f inside γ.

(1) (S 2016 # 5) Let f, g be two holomorphic functions on $|z| < 2$ with f not identically 0. Show that, for $\varepsilon > 0$ sufficiently small, $f - \varepsilon g$ has the same number of zeros inside $|z| = 1$ as f does.

(Note: this problem is not possible as written. You can prove that there exists some $\delta > 0$ such that $f - \varepsilon g$ has the same number of zeros inside $|z| = 1 + \delta$. For all $\lambda < \delta$, there exists some $\varepsilon > 0$ such that the clam is true for the smaller circle.)

Proof. By the Identity Principle, since f is holomorphic and not identically 0, there is some $0 < \delta < 1$ such that f is nonzero on $A = \{z : 1 < |z| \leq 1 + \delta\}$.

(Note: Assume that there are an infinite set of points $W = \{w_i\}_{i\in I}$ in the ball $B_2(0)$ such that $f(w_i) = 0$ for all $i \in I$. Then there must be some set of points in $v_1, v_2, ... \in W$ converging to $v \in W$. The Identity Principle then tells us that f must be 0 on the ball, which is a contradiction. Thus, the zeros of f form a finite set in the ball. Let z_0 be the zero of f with the smallest norm while still satisfying $|z_0| > 1$. If there is no such zero, we can let $z_0 = 2$. If we set $\delta < |z_0| - 1$, then we have that f is nonzero on our annulus.)

Since f and g are continuous, we know that $m = \min_{|z|=1+\delta} |f|$ and $M = \max_{|z|=1+\delta} |g|$ are finite. If $M = 0$, then g is identically zero on the circle of radius $1 + \delta$, so by the Identity Principle, g is 0 in $B_2(0)$, which means $f - \varepsilon g = f$ for all $\varepsilon > 0$, giving us our claim.

Suppose $M > 0$, and let ε, $\frac{m}{M}$, which is clearly greater than 0. Then we have

$$|f - (f - \varepsilon g)| = \varepsilon |g| < \frac{m}{M} M = m \leq |f|$$

on $\{z : |z| = 1 + \delta\}$. By Rouché's Thoerem, f and $f - \varepsilon g$ have the same number of zeros in $\{z : |z| = 1 + \delta\}$. □

(2) (F 2015 # 7) Show that $z^{10} - z^7 + 4z^2 - 1$ has exactly two zeros in $|z| \leq 1$.

Proof. Let $f(z) = 4z^2$ and $g(z) = z^{10} - z^7 + 4z^2 - 1$. On $\{z : |z| = 1\}$, $|f(z)| = 4$ and

$$|f(z) - g(z)| = | - z^{10} + z^7 + 1| \leq 3,$$

so

$$|f(z) - g(z)| < |f(z)|.$$

By Rouché's Theorem, since f has exactly two zeros in $B_1(0)$, counting multiplicities, g does also. □

(3) (S 2015 # 7) Show that $4z^5 - z + 2$ has all its zeros in the unit disk.

Proof. Let $f(z) = 4z^5$ and $g(z) = 4z^5 - z + 2$. On $\{z : |z| = 1\}$, $|f(z)| = 4$ and

$$|f(z) - g(z)| = |z - 2| \leq 3,$$

so

$$|f(z) - g(z)| < |f(z)|.$$

By Rouché's Theorem, since f has exactly five zeros in $B_1(0)$, counting multiplicities, g does also. These are all the zeros of g. □

(4) (F 2014 # 7) Show that $3z^5 - z + 1$ has all its zeros in the unit disk.

Proof. Let $f(z) = 3z^5$ and $g(z) = 3z^5 - z + 1$. On $\{z : |z| = 1\}$, $|f(z)| = 3$ and

$$|f(z) - g(z)| = |z - 1| \leq 2,$$

so

$$|f(z) - g(z)| < |f(z)|.$$

By Rouché's Theorem, since f has exactly five zeros in $B_1(0)$, counting multiplicities, g does also. These are all the zeros of g. □

6. Radius of Convergence

Theorem 6 (Morera's Theorem). *Let $D \subseteq \mathbb{C}$ be open, and f be a continuous, complex-valued function on D. If for every closed, piecewise C^1 curve γ in D*

$$\int_\gamma f(z)dz = 0,$$

then f is holomorphic in D.

(1) (S 2018 # 4) Determine the radius of convergence of the power series for $f(z) = \frac{z}{1-e^z}$ at 0.

Proof. From Cauchy Theory, we know that the power series of f, expanded 0, converges in the largest open ball centered at 0 on which a holomorphic function agrees with f. By inspection, the singularity at 0 is removable:

$$\frac{z}{1-e^z} = \frac{z}{\sum_{n=1}^{\infty} \frac{-z^n}{n!}} = \frac{-1}{\sum_{n=1}^{\infty} \frac{z^{n-1}}{n!}} =: F(z).$$

At $2\pi i$, f is unbounded. Thus, any function agreeing with f cannot be holomorphic at $2\pi i$, so the radius of convergence of f is at most 2π. F defined above is holomorphic for $|z| < 2\pi$ and agrees with f, so the radius of convergence of f is 2π. □

(2) (F 2017 # 2) Determine the radius of convergence of the power series for $\log(z)$ centered at $-3+4i$.

Proof. From Cauchy Theory, we know that the power series of $\log(z)$, expanded $-3+4i$, converges in the largest open ball centered at $-3+4i$ on which a holomorphic function agrees with $\log(z)$.

Consider

$$L(z) = \log(-3+4i) + \int_\gamma \frac{dw}{w}$$

where γ is the line segment from $-3+4i$ to z.

By inspection $L(z)$ is holomorphic on $\mathbb{C} \setminus \{(3-4i)t : t \geq 0\}$. The largest open disk in this set, centered at $-3+4i$ is $B_5(-3+4i)$ and $L(z)$ agrees with $\log(z)$ on this disk. Since $\log(z)$ is unbounded at 0, we know that the radius of convergence is at most 5, giving us that the radius of convergence of the power series is 5. □

(3) (S 2017 # 4) Determine the radius of convergence of the power series for $f(z) = \frac{z^2}{1-\cos(z)}$ at 0.

Proof. From Cauchy Theory, we know that the power series of f, expanded 0, converges in the largest open ball centered at 0 on which a holomorphic function agrees with f.

By inspection, the singularity at 0 is removable:

$$\frac{z^2}{1-\cos(z)} = \frac{z^2}{\sum_{n=1}^{\infty}(-1)^{n+1}\frac{z^{2n}}{(2n)!}} = \frac{1}{\sum_{n=1}^{\infty}(-1)^{n+1}\frac{z^{2n-2}}{(2n)!}} := F(z).$$

At 2π, f is unbounded. Thus, any function agreeing with f cannot be holomorphic at 2π, so the radius of convergence of f is at most 2π. F defined above is holomorphic for $|z| < 2\pi$ and agrees with f, so the radius of convergence of f is 2π. □

(4) (F 2016 # 3) Determine the radius of convergence of the power series for $f(z) = \frac{z}{\sin(z)}$ at 0.

Proof. From Cauchy Theory, we know that the power series of f, expanded 0, converges in the largest open ball centered at 0 on which a holomorphic function agrees with f.

By inspection, the singularity at 0 is removable:

$$\frac{z}{\sin(z)} = \frac{z}{\sum_{n=0}^{\infty}(-1)^{n+1}\frac{z^{2n+1}}{(2n+1)!}} = \frac{1}{\sum_{n=0}^{\infty}(-1)^{n+1}\frac{z^{2n}}{(2n+1)!}} := F(z).$$

At π, f is unbounded. Thus, any function agreeing with f cannot be holomorphic at π, so the radius of convergence of f is at most π. F defined above is holomorphic for $|z| < \pi$ and agrees with f, so the radius of convergence of f is π. □

(5) (S 2016 # 2) Determine the radius of convergence of the power series for $\log(z)$ centered at $-4+3i$.

Proof. From Cauchy Theory, we know that the power series of $\log(z)$, expanded $-4+3i$, converges in the largest open ball centered at $-4+3i$ on which a holomorphic function agrees with $\log(z)$.

Consider

$$L(z) = \log(-4+3i) + \int_\gamma \frac{dw}{w}$$

where γ is the line segment from $-3+4i$ to z.

By inspection $L(z)$ is holomorphic on $\mathbb{C} \setminus \{(4-3i)t : t \geq 0\}$. The largest open disk in this set, centered at $-4+3i$ is $B_5(-4+3i)$ and $L(z)$ agrees with $\log(z)$ on this disk. Since $\log(z)$ is unbounded at 0, we know that the radius of convergence is at most 5, giving us that the radius of convergence of the power series is 5. □

(6) (F 2015 # 2) Determine the radius of convergence of the power series for $\log(z)$ centered at $-4+3i$.

Proof. See above. □

(7) (S 2015 # 4) Determine the radius of convergence of the power series for $\sqrt{z}$ expanded at $-4+3i$.

Proof. From Cauchy Theory, we know that the power series of $\sqrt{z}$, expanded $-4+3i$, converges in the largest open ball centered at $-4+3i$ on which a holomorphic function agrees with $\sqrt{z}$.

Consider

$$L(z) = \log(-4+3i) + \int_\gamma \frac{dw}{w}$$

where γ is the line segment from $-4+3i$ to z.

By inspection, $L(z)$ is holomorphic on $D = \mathbb{C} \setminus \{(4-3i)t : t \geq 0\}$. Let $S(z) = e^{\frac{1}{2}L(z)}$. By construction, $S(z)$ is holomorphic everywhere $L(z)$ is. The largest open disk in D, centered at $-4+3i$, is $B_5(-4+3i)$ and $S(z)$ agrees with $\sqrt{z}$ on this disk, so the radius of convergence of the power series is at least 5.

Now, suppose that $\sqrt{z}$ was holomorphic at 0. Then there is a power series expanded at zero:

$$\sqrt{z} = \sum_{n=0}^{\infty} c_n z^n.$$

This means that

$$z = c_0 + 2c_0c_1z + \cdots,$$

meaning that $c_0 = 0$ and $2c_0c_1 = 1$, which is a contradiction, meaning that $\sqrt{z}$ is not holomorphic at 0. Thus the radius of convergence is 5. □

(8) (F 2014 # 3) What is the radius of convergence of the power series for $\log(z)$ centered at $-4+3i$.

Proof. See above. □

7. Holomorphic functions

Theorem 7 (Cauchy-Reimann Equations)**.** *Let x, y be real-variables, u, v be real-valued functions, and $f : \mathbb{C} \to \mathbb{C}$ such that*

$$f(x+iy) = u(x+iy) + iv(x+iy).$$

Then f is holomorphic at $x_0 + iy_0$ if and only if

$$\frac{\partial u}{\partial x}(x_0+iy_0) = \frac{\partial v}{\partial y}(x_0+iy_0),$$

and

$$\frac{\partial u}{\partial y}(x_0+iy_0) = -\frac{\partial v}{\partial x}(x_0+iy_0).$$

Theorem 8 (Cauchy's Formula)**.** *If f is complex-differentiable, it is holomorphic. For f holomorphic on some neighborhood U of z, and*

$$f^{(n)}(z) = \frac{n!}{2\pi i}\int_\gamma \frac{f(w)}{(w-z)^{n+1}}dw,$$

for any closed path $\gamma \subset U$ about z.

Theorem 9 (Liouville's Theorem)**.** *A bounded entire function is constant.*

Proposition 1 (Reflection Principle)**.** *Let Ω be an open subset of the complex upper half-plane, $\overline{\Omega} \cap \mathbb{R} = [a, b] \subseteq \mathbb{R}$, with $a < b$. Then any holomorphic f on Ω extending to a continuous function on $\Omega \cup (a, b)$, real-valued on (a, b), extends to a holomorphic function $\tilde{f}$ on*

$$\Omega \cup (a, b) \cup \{\bar{z} : z \in \Omega\}.$$

The extension is defined on the reflected image by

$$\tilde{f}(\bar{z}) = \overline{f(z)}, \ (z \in \Omega).$$

Theorem 10 (Open Mapping Theorem)**.** *A non-constant holomorphic function maps open sets to open sets.*

(1) (S 2018 # 2) Let f be an entire function such that $\Re f(z)$ is bounded. Show that f is constant.

Proof. Let $C > 0$ such that $|f(z)| \leq C$ for all $z \in \mathbb{C}$.

Since f is entire, we know that $e^{f(z)}$ is an entire function. We know that $|e^{f(z)}| \leq e^C$, which, by Liouville's Theorem, means that $e^{f(z)}$ is constant. Since f is entire, we know that f must also be constant. □

(2) (F 2017 # 3) Show that $\frac{\sin(\sqrt{z})}{\sqrt{z}}$ is entire.

Proof. Let $z_0 \in \mathbb{C} \setminus \{0\}$. Let $L_{z_0}(z)$ be defined by

$$L_{z_0}(z) = \log(z_0) + \int_{z_0}^{z} \frac{dw}{w},$$

where the integral is taken over the line segment from z_0 to z. This is well defined and holomorphic on $B_{|z_0|}(z_0)$.

Thus, there is some holomorphic square root function $S_{z_0}(z) = e^{\frac{1}{2}L_{z_0}(z)}$ on $B_{|z_0|}(z_0)$.

We see that on $B_{|z_0|}(z_0)$,

$$\begin{aligned}\frac{\sin(S_{z_0}(z))}{S_{z_0}(z)} &= \frac{1}{S_{z_0}(z)} \sum_{k=0}^{\infty} (-1)^k \frac{(S_{z_0}(z))^{2k+1}}{(2k+1)!} \\ &= \sum_{k=0}^{\infty} (-1)^k \frac{(S_{z_0}(z))^{2k}}{(2k+1)!} \\ &= \sum_{k=0}^{\infty} (-1)^k \frac{z^k}{(2k+1)!}.\end{aligned}$$

This sum converges everywhere, and has no dependence on z_0. This means the singularity of $\frac{\sin(\sqrt{z})}{\sqrt{z}}$ at 0 is removable. Let

$$g(z) := \sum_{k=0}^{\infty} (-1)^k \frac{z^k}{(2k+1)!}.$$

g is holomorphic and agrees with $\frac{\sin(\sqrt{z})}{\sqrt{z}}$ everywhere, giving us that $\frac{\sin(\sqrt{z})}{\sqrt{z}}$ is entire. □

(3) (F 2017 # 4) Show that a holomorphic function f on a non-empty open set, $\Omega \subseteq \mathbb{C}$, with $|f|$ constant is itself constant.

Proof. There exists some $C \geq 0$ such that $f(\Omega) \subseteq \{C \cdot e^{it} : t \in \mathbb{R}\}$. This means that $f(\Omega)$ is not open. By the Open Mapping Theorem, we then have that f is constant.

□

(4) (F 2017 # 6) Let f be holomorphic in the open upper half-plane, with $|f(x + iy)| \leq y$ for $y > 0$. Show that f is constant.

Proof. f extends to a continuous function on $\mathfrak{h} \cup \mathbb{R}$ which takes the value 0 on $\mathbb{R}$. By the Reflection Principle, f extends to a holomorphic function $\tilde{f}$ on $\mathbb{C}$ such that $\tilde{f} = f$ on $\mathfrak{h}$, $\tilde{f} = 0$ on $\mathbb{R}$, and $\tilde{f}(\overline{z}) = \overline{f(z)}$ for all $z \in \mathfrak{h}$. We then have that $|\tilde{f}(x+iy)| \leq |y|$ for all $x, y \in \mathbb{R}$.

Since $\tilde{f}$ is entire, it has a power series expansion centered at 0, meaning

$$\tilde{f}(z) = \sum_{n=0}^{\infty} \frac{\tilde{f}^{(n)}(0)}{n!} z^n$$

for all $z \in \mathbb{C}$. For any $R > 0$, we have

$$\begin{aligned} |\tilde{f}^{(n)}(0)| &= \left| \frac{n!}{2\pi i} \int_{|w|=R} \frac{\tilde{f}(w)}{(w-0)^{n+1}} dw \right| \\ &\leq \frac{n!}{2\pi} \int_0^{2\pi} \frac{|\tilde{f}(Re^{it})| \cdot |Re^{it}|}{R^{n+1}} dt \\ &\leq n! \frac{R}{R^n}. \end{aligned}$$

Since this is true for all $R > 0$, we know that $f^{(n)}(0) = 0$ for $n > 1$. Thus, $\tilde{f}$ is a polynomial of degree at most 1. However, since $\tilde{f} = 0$ on $\mathbb{R}$, $\tilde{f}$ cannot be a linear polynomial, so $\tilde{f}$ is constant. This gives us that f is constant. ☐

(5) (S 2017 # 2) Show that an $\mathbb{R}$-valued holomorphic function f is constant.

Proof. Let $\Omega \subseteq \mathbb{C}$ be an open set, and let $f : \Omega \to \mathbb{R}$ be holomorphic. Since any subset $\mathbb{R}$ is not open in $\mathbb{C}$, the Open Mapping theorem tells us f must be constant. ☐

(6) (S 2017 # 5) Let f be an entire function such that $|f(z)| \leq 1 + \sqrt{|z|}$ for all $z \in \mathbb{C}$. Show that f is constant.

Proof. Since f is entire, is has a power series expansion centered at 0, meaning

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} z^n$$

for all $z \in \mathbb{C}$. For any $R > 0$, we have

$$\begin{aligned} |f^{(n)}(0)| &= \left| \frac{n!}{2\pi i} \int_{|w|=R} \frac{f(w)}{(w-0)^{n+1}} dw \right| \\ &\leq \frac{n!}{2\pi} \int_0^{2\pi} \frac{|f(Re^{it})| \cdot |Re^{it}|}{R^{n+1}} dt \\ &\leq n! \frac{1\sqrt{R}}{R^n}. \end{aligned}$$

Since this is true for all $R > 0$, we know that $f^{(n)}(0) = 0$ for $n > 0$. Thus, f is constant. ☐

(7) (F 2016 # 4) Show that a holomorphic function f with $|f(z)| = 1$ for all z is constant.

Proof. Let $\Omega \subseteq \mathbb{C}$ be an open set, and let $f : \Omega \to S$, with $S = \{z \in \mathbb{C} : |z| = 1\}$ be holomorphic. Since any subset $\mathbb{R}$ is not open in $\mathbb{C}$, the Open Mapping theorem tells us f must be constant. ☐

(8) (F 2016 # 6) Let f be holomorphic, bounded in the open upper half-plane $\mathfrak{h}$, and real-valued on $\mathbb{R}$. Show that f is constant.

Proof. (I think you need to assume that f is continuous on $\mathfrak{h} \cup \mathbb{R}$) By the Reflection Principle, f extends to a holomorphic function $\tilde{f}$ on $\mathbb{C}$ such that $\tilde{f} = f$ on $\mathfrak{h} \cup \mathbb{R}$ and $\tilde{f}(\overline{z}) = \overline{f(z)}$ for all $z \in \mathfrak{h}$. Due to continuity near $\mathbb{R}$, f is bounded on $\mathbb{R}$. Since f is bounded in $\mathfrak{h}$, $\tilde{f}$ is bounded on $\mathbb{C}$. Since $\tilde{f}$ is entire and bounded, it is constant. Thus, f is constant. ☐

(9) (S 2016 # 4) Show that a holomorphic function f on $\mathbb{C}$ satisfying $|f(z)| \leq \sqrt{|z|}$ for all $z \in \mathbb{C}$ is 0.

Proof. Since f is entire, is has a power series expansion centered at 0, meaning

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} z^n$$

for all $z \in \mathbb{C}$. For any $R > 0$, we have

$$\begin{aligned} |f^{(n)}(0)| &= \left| \frac{n!}{2\pi i} \int_{|w|=R} \frac{f(w)}{(w-0)^{n+1}} dw \right| \\ &\leq \frac{n!}{2\pi} \int_0^{2\pi} \frac{|f(Re^{it})| \cdot |Re^{it}|}{R^{n+1}} dt \\ &\leq n! \frac{\sqrt{R}}{R^n}. \end{aligned}$$

Since this is true for all $R > 0$, we know that $f^{(n)}(0) = 0$ for $n > 0$. Thus, f is constant. Since $|f(0)| \leq 0$, we have that $f = 0$. □

(10) (F 2015 # 3) Show that a real-valued holomorphic function is constant.

Proof. See above. □

(11) (F 2015 # 6) Let f be an entire function such that $f(z+1) = f(z) = f(z+i)$ for all z. Show that f is constant.

Proof. Since f is entire, it must be bounded in $S = \{z : 0 \leq \text{Re}(z) \leq 1, 0 \leq \text{Im}(z) \leq 1\}$. For any $w \in \mathbb{C}$, there exists some $z \in S$ such that $z = w + n + im$ for some $n, m \in \mathbb{Z}$. Inductively, we can see that this means that $f(z) = f(w)$. This gives us that f is bounded on $\mathbb{C}$. By Liouville's Theorem, we then have that f is constant. □

(12) (S 2015 # 6) Show that a holomorphic function f on $\mathbb{C}$ satisfying $|f(z)| \leq \sqrt{1+|z|}$ for all $z \in \mathbb{C}$ is constant.

Proof. Since f is entire, is has a power series expansion centered at 0, meaning

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} z^n$$

for all $z \in \mathbb{C}$. For any $R > 0$, we have

$$\begin{aligned} |f^{(n)}(0)| &= \left| \frac{n!}{2\pi i} \int_{|w|=R} \frac{f(w)}{(w-0)^{n+1}} dw \right| \\ &\leq \frac{n!}{2\pi} \int_0^{2\pi} \frac{|f(Re^{it})| \cdot |Re^{it}|}{R^{n+1}} dt \\ &\leq n! \frac{\sqrt{1+R}}{R^n}. \end{aligned}$$

Since this is true for all $R > 0$, we know that $f^{(n)}(0) = 0$ for $n > 0$. Thus, f is constant.

□

(13) (F 2014 # 6) Classify the holomorphic functions f on $\mathbb{C}$ which satisfy $|f(z)| \leq (1+|z|)^2$ for all $z \in \mathbb{C}$.

Proof. Since f is entire, is has a power series expansion centered at 0, meaning

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} z^n$$

for all $z \in \mathbb{C}$. For any $R > 0$, we have

$$\begin{aligned} |f^{(n)}(0)| &= \left| \frac{n!}{2\pi i} \int_{|w|=R} \frac{f(w)}{(w-0)^{n+1}} dw \right| \\ &\leq \frac{n!}{2\pi} \int_0^{2\pi} \frac{|f(Re^{it})| \cdot |Re^{it}|}{R^{n+1}} dt \\ &\leq n! \frac{(1+R)^2}{R^n}. \end{aligned}$$

Since this is true for all $R > 0$, we know that $f^{(n)}(0) = 0$ for $n > 2$.

Thus, if f is entire with $|f(z)| \leq (1 + |z|)^2$ for all $z \in \mathbb{C}$, f is a polynomial of degree at most 2. □

8. Elliptic Curves

We need compact surfaces to use Riemann-Hurwitz. Fortunately, there is an inclusion of $\mathbb{C} \to \mathbb{CP}$ given by $z \mapsto \begin{pmatrix} z \\ 1 \end{pmatrix}$, and we can consider $\begin{pmatrix} z \\ 0 \end{pmatrix}$ to be the point at infinity. Thus, for the rest of this work, despite us saying that the points are in the projective space, we will abuse notation and work with them as if they were in $\mathbb{C} \cup \{\infty\}$.

Theorem 11. *For a degree n ramified cover $\pi : Y \to X$ of compact, connected, Riemann surfaces, the genus g_Y of Y is related to the genus g_X of X by*

$$2g_Y - 2 = n \cdot (2g_X - 2) + \sum_{\textit{ramified } z} (e_z - 1),$$

where the sum is over points $z \in Y$ ramified over X and e_z is the ramification index of z.

Note, we are covering X. In the cases we will be dealing with, where we map (x, y) to x, the degree of the cover will be the highest degree of y in the polynomial.

Let Y be the curve in $\mathbb{CP}^1 \times \mathbb{CP}^1$ define by $F(x, y) = 0$, for some polynomial F. Let $\pi : Y \to \mathbb{CP}^1$ be defined by $\pi(x, y) = x$. At point $(x_0, y_0) \in Y$, let $F(x, y)$ be rewritten as a monic polynomial of $(y - y_0)$ with coefficients in $\mathbb{C}(x)$ (note that $c_n = 1$):

$$\sum_{j=0}^{n} c_{n-j}(x)(y - y_0)^{n-j}.$$

Let $\text{ord}_{x_0} f(x)$ be the **order of vanishing** of rational function f at x_0, i.e. if

$$f(x) = (x - x_0)^n \frac{P(x)}{Q(x)}$$

such that P, Q are polynomials, with $P(x_0) \neq 0$ and $Q(x_0) \neq 0$, then $\text{ord}_{x_0} f(x) = n$.

For $j \in \{0, 1, ..., n\}$, let $k(j) = \text{ord}_{x_0} c_{n-j}(x)$, with $k(j) = \infty$ if $c_{n-j} = 0$. Let P be the maximum piecewise-linear, convex function on $[0, n]$ such that $P(j) \leq k(j)$ for all $j \in \{0, ..., n\}$.

Let $j_1 < ... < j_m$ be the points at which $P(j_i) = k(j_i)$. The line segments

$$l_i = \text{ line segment connecting } (j_i, P(j_i)) \text{ and } (j_{i+1}, P(j_{i+1})$$

form the **Newton Polygon** attached to f.

Proposition 2. *Every segment that has a rise n and run m that are relatively prime at point (x_0, y_0) is ramified with index m over x_0.*

(1) (S 2018 # 6) Prove that $w^2 = z^4 + 1$ defines an elliptic curve.

Proof. This ramified covering of $\mathbb{CP}^1$ by $(z, w) \mapsto z$ is of degree 2. There are 2 distinct local square root functions w over all z except the four zeros of $z^4 + 1$, since there is no square root function on a neighborhood of 0.

Consider the Newton polygon of $w^2 - (z^4 + 1)$ with order taken with respect to z. The coefficients have order 0, ∞, and 1 at each of the four zeros of $z^4 + 1$. Since these polygons have slope $\frac{1}{2}$, the corresponding points have ramification index 2.

To determine the ramification above ∞, we use coordinates $\frac{1}{z}$, $\frac{1}{w}$ in place of z, w and look near 0:

$$\begin{aligned} \frac{1}{w^2} &= \frac{1}{z^4} + 1 \\ z^4 &= w^2 + w^2 z^4 \\ \frac{z^4}{1 + z^4} &= w^2. \end{aligned}$$

Near 0, there are two distinct square roots of $\frac{1}{1+z^4}$, so there are two distinct holomorphic functions $w = \frac{z^2}{\sqrt{1+z^4}}$ and $w = \frac{-z^2}{\sqrt{1+z^4}}$ near 0. Thus, there is no ramification above ∞.

By the Riemann-Hurwitz Theorem, the genus g of this ramified cover is determined by

$$2g - 2 = n \cdot (2g_{\mathbb{CP}^1} - 2) + \sum_{\text{ramified } y_0} (e_{y_0} - 1),$$

giving us

$$2g - 2 = 2 \cdot (2 \cdot 0 - 2) + \sum_{y_0 \in \{\omega, \omega^3, \omega^5, \omega^7\}} (2 - 1) = 0,$$

where $\omega = e^{\pi i/4}$. Thus, we indeed have that $g = 1$, meaning that the curve described by $w^2 = z^4 + 1$ is indeed elliptic. □

(2) (F 2016 # 9) Prove that $w^2 = z^4 + 1$ defines an elliptic curve, that is, has genus 1.

Proof. See above. □

(3) (S 2016 # 8) Show that the curve $\{(z, w) \in \mathbb{C}^2 : z^3 + w^3 = 1$ (with points added at infinity as needed) is an elliptic curve, that is, has genus $g = 1$.

Proof. This problem is equivalent to showing that the same curve considered on $(\mathbb{CP}^1)^2$ has genus 1.

This ramified covering of $\mathbb{CP}^1$ by $(z, w) \to z$ is of degree 3. There are 3 distinct local cubed root functions w over all $z \in \mathbb{C}$ except the the zeros of $1 - z^3$, ($1, \omega, \omega^2$, where $\omega = e^{2\pi i/3}$) since there is no cube root function on a neighborhood of 0.

Consider the Newton polygon of $w^3 - (1 - z^3)$ with order taken with respect to z. The coefficients have order 0, ∞, ∞, and 1 at each of the three zeros of $1 - z^3$. Since these polygons have slope $\frac{1}{3}$, the corresponding points have ramification index 3.

To determine the ramification above ∞, we use coordinates $\frac{1}{z}$, $\frac{1}{w}$ in place of z, w and look near 0:

$$\begin{aligned}\frac{1}{w^3} &= 1-\frac{1}{z^3}\\ z^3 &= w^3z^3-w^3\\ \frac{z^3}{z^3-1} &= w^3.\end{aligned}$$

Near 0, there are three distinct cubed roots of $\frac{1}{z^3-1}$, so there are three distinct holomorphic functions $w=\frac{z}{(z^3-1)^{1/3}}$, $w=\frac{\omega z}{(z^3-1)^{1/3}}$, and $w=\frac{\omega^2 z}{(z^3-1)^{1/3}}$ near 0. Thus, there is no ramification above ∞.

By the Riemann-Hurwitz Theorem, the genus g of this ramified cover is determined by

$$2g-2=n\cdot(2g_{\mathbb{CP}^1}-2)+\sum_{\text{ramified } y_0}(e_{y_0}-1),$$

giving us

$$2g-2=3\cdot(2\cdot 0-2)+\sum_{y_0\in\{1,\omega,\omega^2\}}(3-1)=0.$$

Thus, we indeed have that $g=1$. □

9. Change of Coordinates

Definition 1. *A* ***linear fractional transformation*** *is a transformation of the form $f:\mathbb{C}^*\to\mathbb{C}^*$, such that*

$$f(z)=\frac{az+b}{cz+d},$$

where, $a,b,c,d\in\mathbb{C}$, $ad-bc\neq 0$, $f(-\frac{d}{c})=\infty$, and $f(\infty)=\frac{a}{c}$.

(1) (F 2017 # 8) Make a change of coordinates to put the elliptic curve $w^2=z^4+1$ into the (essential) Weierstrass form $y^2=x^3+bx+c$.

Proof. We take a linear fractional transformation f sending ∞ to a zero of z^4+1, such as $z\to\frac{z\cdot e^{\frac{\pi i}{4}}+1}{z}$. This gives us the equation

$$\begin{aligned}w^2 &= \left(\frac{z\cdot e^{\frac{\pi i}{4}}+1}{z}\right)^4+1\\ z^4w^2 &= 4z^3\cdot e^{\frac{3\pi i}{4}}+6z^2\cdot e^{\frac{\pi i}{2}}+4z\cdot e^{\frac{\pi i}{4}}+1.\end{aligned}$$

Transforming w to $\frac{2y\cdot e^{\frac{3\pi i}{8}}}{z^2}$ then gives us

$$y^2=z^3+\frac{3}{2}z^2\cdot e^{\frac{-\pi i}{4}}+z\cdot e^{\frac{-\pi i}{2}}+\frac{e^{\frac{-3\pi i}{4}}}{4}.$$

Finally, taking the transformation $z\to(x-\frac{1}{2}e^{\frac{-\pi i}{4}})$ gives us

$$\begin{aligned}y^2 &= \left(x-\frac{1}{2}e^{\frac{-\pi i}{4}}\right)^3+\frac{3}{2}e^{\frac{-\pi i}{4}}\cdot\left(x-\frac{1}{2}e^{\frac{-\pi i}{4}}\right)^2+e^{\frac{-\pi i}{2}}\cdot\left(x-\frac{1}{2}e^{\frac{-\pi i}{4}}\right)+\frac{e^{\frac{-3\pi i}{4}}}{4}\\ &= x^3+x\frac{3}{4}e^{\frac{-\pi i}{2}}-\frac{1}{8}e^{\frac{-3\pi i}{4}}-x\frac{3}{2}e^{\frac{-\pi i}{2}}+\frac{3}{8}e^{\frac{-3\pi i}{4}}+xe^{\frac{-\pi i}{2}}-e^{\frac{-3\pi i}{4}}+\frac{e^{\frac{-3\pi i}{4}}}{4}\\ &= x^3+x\frac{1}{4}e^{\frac{-\pi i}{2}}-\frac{1}{2}e^{\frac{-3\pi i}{4}},\end{aligned}$$

giving us our Weierstrass form. □

(2) (S 2017 # 8) Make a change of coordinates to put the elliptic curve $w^2 = z^4 + 1$ into the (essential) Weierstrass form $y^2 = x^3 + bx + c$.

Proof. See above. □

10. Conformal Mapping

Definition 2. *A complex-valued function f on a non-empty open $U \subseteq \mathbb{C}$ is* ***conformal*** *if it preserves angles, i.e. for any two smooth parameterized curves $\gamma : [a, b] \to \mathbb{C}$, $\lambda : [c, d] \to \mathbb{C}$ with $\gamma(a) = \lambda(c)$, the angle between $\gamma'(a)$ and $\lambda'(c)$ is equal to the angle between $(f \circ \gamma)'(a)$ and $(f \circ \lambda)'(c)$.*

Proposition 3. *A holomorphic function f is conformal at z if $f'(z) \neq 0$.*

Theorem 12. *The collection of lines and circles in $\mathbb{C} \cup \{\infty\}$ is stabilized by linear fractional transformations, and is acted upon transitively by them.*

(1) (F 2015 # 4) Give an explicit conformal mapping from $X = \{z : |z - 1| < \sqrt{2}, |z + 1| < \sqrt{2}\}$ to the unit disk $\{z : |z| < 1\}$.

Proof. Let $\Omega = \{z = re^{i\theta} : r > 0, -\frac{\pi}{4} < \theta < \frac{\pi}{4}\}$, Q be the first quadrant, $\mathfrak{h}$ be the upper half-plane, and $\mathbb{D}$ be the unit disk.

- The Cayley map $h : z \mapsto \frac{z+i}{iz+1}$ is a Möbius transformation and is thus conformal. X is the intersection of the disks of radius $\sqrt{2}$ centered at ± 1. h is a linear fractional transformation, so it preserves circles/lines, and since $-i \mapsto 0$, $0 \mapsto i$, $i \mapsto \infty$, $1 - \sqrt{2} \mapsto \frac{\sqrt{2}}{2}(i - 1) \in \{\mathrm{Re}(z) = -\mathrm{Im}(z)\}$, and $\sqrt{2} - 1 \mapsto \frac{\sqrt{2}}{2}(i + 1) \in \{\mathrm{Re}(z) = \mathrm{Im}(z)\}$, we have that $h(X) = \Omega$, so h is bijective on X.
- The map $p : z \mapsto ze^{-\pi i/4}$ is holomorphic, with non-vanishing derivative on Ω, so p is conformal. Since $p(\Omega) = Q$, it is also clearly bijective.
- The map $f : z \mapsto z^2$ is holomorphic, with non-vanishing derivative on Q, so f is conformal. Since $f(Q) = \mathfrak{h}$, it is also clearly bijective.
- The inverse Cayley map $g : z \mapsto \frac{z-i}{-iz+1}$ is a Möbius transformation and is thus conformal. g is a linear fractional transformation, so it preserves circles/lines, and since $-1 \mapsto -1$, $0 \mapsto -i$, $1 \mapsto 1$, and $i \mapsto 0$, we have that $g(\mathfrak{h}) = \mathbb{D}$, so g is bijective on $\mathfrak{h}$.

Thus, the map $g \circ f \circ p \circ h : X \to \mathbb{D}$ is conformal and bijective.

□

(2) (S 2015 # 3) Give an explicit conformal mapping from the half disk $\{z : |z| < 1,\ \mathrm{Re}(z) > 0\}$ to the unit disk $\{z : |z| < 1\}$.

Proof. Let Ω be the half disk, Q be the first quadrant, $\mathfrak{h}$ be the upper half-plane, and $\mathbb{D}$ be the unit disk.

- The Cayley map $h : z \mapsto \frac{z+i}{iz+1}$ is a Möbius transformation and is thus conformal. h is a linear fractional transformation, so it preserves circles/lines, and since $-i \mapsto 0$, $0 \mapsto i$, $1 \mapsto 1$, and $i \mapsto \infty$, we have that $h(\Omega) = Q$, so h is bijective on Ω.
- The map $f : z \mapsto z^2$ is holomorphic, with non-vanishing derivative on Q, so f is conformal. Since $f(Q) = \mathfrak{h}$, it is also clearly bijective.
- The inverse Cayley map $g : z \mapsto \frac{z-i}{-iz+1}$ is a Möbius transformation and is thus conformal. g is a linear fractional transformation, so it preserves circles/lines, and since $-1 \mapsto -1$, $0 \mapsto -i$, $1 \mapsto 1$, and $i \mapsto 0$, we have that $g(\mathfrak{h}) = \mathbb{D}$, so g is bijective on $\mathfrak{h}$.

Thus, the map $g \circ f \circ h : \Omega \to \mathbb{D}$ is conformal and bijective. □

(3) (F 2014 # 2) Give an explicit conformal mapping that gives a bijection of the first quadrant with the unit disk.

Proof. Let Q by the first quadrant, $\mathfrak{h}$ be the upper half-plane, and $\mathbb{D}$ be the unit disk.

- The map $f : z \mapsto z^2$ is holomorphic, with non-vanishing derivative on Q, so f is conformal. Since $f(Q) = \mathfrak{h}$, it is also clearly bijective.

- The inverse Cayley map $g : z \mapsto \frac{z-i}{-iz+1}$ is a Möbius transformation and is thus conformal. g is a linear fractional transformation, so it preserves circles/lines, and since $-1 \mapsto -1$, $0 \mapsto -i$, $1 \mapsto 1$, and $i \mapsto 0$, we have that $g(\mathfrak{h}) = \mathbb{D}$, so g is bijective on $\mathfrak{h}$.

Thus, the map $g \circ f : Q \to \mathbb{D}$ is conformal and bijective. □

11. Holomorphic Roots

(1) (S 2018 # 5) Show that there is a holomorphic function f on the region $1 < |z| < 2$ such that $f(z)^2 = (z^2-1)(z^2-4)$.

Proof. Let $H = \{z : 1 < |z| < 2\}$, and $g(z) = (z^2-1)(z^2-4)$. Let $z_0 \in H$. For all $z \in H$, let

$$L(z) = \log(g(z_0)) + \int_\gamma \frac{g'(w)dw}{g(w)}$$

where γ is a path in H from z_0 to z.

Let γ_1 and γ_2 be paths in H from z_0 to z, and let γ be the path that goes from z_0 to z on γ_1 and from z to z_0 on γ_2. If $\overline{B_1(0)}$ is not inside γ, then

$$0 = \int_\gamma \frac{g'(w)dw}{g(w)} = \int_{\gamma_1} \frac{g'(w)dw}{g(w)} - \int_{\gamma_2} \frac{g'(w)dw}{g(w)}.$$

If $\overline{B_1(0)}$ is inside of γ, then by the Residue Theorem, we have

$$\begin{aligned}\int_{\gamma_1} \frac{g'(w)dw}{g(w)} - \int_{\gamma_2} \frac{g'(w)dw}{g(w)} &= \int_\gamma \frac{g'(w)dw}{g(w)} \\ &= 2\pi i \cdot m \left(\text{Res}_{z=1}\frac{g'(w)}{g(w)} + \text{Res}_{z=-1}\frac{g'(w)}{g(w)}\right) \\ &= 4\pi i \cdot m.\end{aligned}$$

So, for any two paths γ_1 and γ_2 from z_0 to z in H, we have

$$\int_{\gamma_1} \frac{g'(w)dw}{g(w)} = 4\pi i \cdot n + \int_{\gamma_2} \frac{g'(w)dw}{g(w)}$$

for some $n \in \mathbb{Z}$.

Let $f(z) = Ce^{\frac{1}{2}L(z)}$, for an appropriate constant C. We see that f is independent of the path of our integral, so it is well defined.

Let $z \in H$, and $\delta > 0$ such that $B_\delta(z) \subset H$, and let $|h| < \delta$. Let p_h be the line segment from z to $z+h$. Since f is independent of paths, we have

$$\begin{aligned}f(z+h) &= Ce^{\frac{1}{2}\left(\log(z_0) + \int_\gamma \frac{g'(w)dw}{g(w)} + \int_{p_h} \frac{g'(w)dw}{g(w)}\right)} \\ &= f(z)e^{\frac{1}{2}\int_{p_h} \frac{g'(w)dw}{g(w)}}.\end{aligned}$$

This gives us

$$\begin{aligned}\frac{f(z+h)-f(z)}{h} &= \frac{f(z)}{h}(e^{\frac{1}{2}\int_{p_h} \frac{g'(w)dw}{g(w)}} - 1) \\ &= \frac{f(z)}{h}\sum_{n=1}^\infty \frac{\left(\frac{1}{2}\int_{p_h} \frac{g'(w)dw}{g(w)}\right)^n}{n!} \\ &= \frac{f(z)}{2}\frac{\left(\int_{p_h} \frac{g'(w)dw}{g(w)}\right)}{h}\sum_{n=1}^\infty \frac{\left(\frac{1}{2}\int_{p_h} \frac{g'(w)dw}{g(w)}\right)^{n-1}}{n!}\end{aligned}$$

so

$$\lim_{h\to 0}\frac{f(z+h)-f(z)}{h}=\frac{f(z)}{2}\frac{g'(z)}{g(z)}.$$

We see that f is differentiable on H, so it is holomorphic.

Finally, we have that for $z\in H$,

$$\frac{d}{dz}\frac{g(z)}{f(z)^2}=\frac{-2}{2}f(z)\frac{g'(z)}{g(z)}\frac{1}{f(z)^3}g(z)+\frac{g'(z)}{f(z)^2}=0,$$

so $\frac{g(z)}{f(z)^2}$ is a constant. By choosing the appropriate C, we have that $f(z)^2=g(z)$.

□

(2) (S 2017 # 6) Show that there is a holomorphic function f on the region $|z|>2$ such that $f(z)^4=z^4+z+1$.

Proof. Let $g(z)=z^4+z+1$. For $z\geq 2$, we have that $|z^4|>|z|+1$, so by the triangle inequality, we have that $|z^4+z+1|>0$, so all zeros of $g(z)$ are in $B_2(0)$.

Let $H=\mathbb{C}\setminus B_2(0)$, and let $z_0\in H$. for all $z\in H$, let

$$L(z)=\log(g(z_0))+\int_\gamma\frac{g'(w)dw}{g(w)}$$

where γ is a path in H from z_0 to z.

Let γ_1 and γ_2 be paths in H from z_0 to z, and let γ be the path that goes from z_0 to z on γ_1 and from z to z_0 on γ_2. If $\overline{B_2(0)}$ is not inside γ, then

$$0=\int_\gamma\frac{g'(w)dw}{g(w)}=\int_{\gamma_1}\frac{g'(w)dw}{g(w)}-\int_{\gamma_2}\frac{g'(w)dw}{g(w)}.$$

If $\overline{B_2(0)}$ is inside of γ, then by the Residue Theorem, we have

$$\begin{aligned}\int_{\gamma_1}\frac{g'(w)dw}{g(w)}-\int_{\gamma_2}\frac{g'(w)dw}{g(w)}&=\int_\gamma\frac{g'(w)dw}{g(w)}\\&=2\pi i\cdot m\left(\sum_{i=1}^{4}\text{Res}_{z=z_i}\frac{g'(w)}{g(w)}\right)\\&=8\pi i\cdot m\end{aligned}$$

for some $m\in\mathbb{Z}$, where z_1,z_2,z_3,z_4 are the zeros of g.

So, for any two paths γ_1 and γ_2 from z_0 to z in H, we have

$$\int_{\gamma_1}\frac{g'(w)dw}{g(w)}=8\pi i\cdot n+\int_{\gamma_2}\frac{g'(w)dw}{g(w)}$$

for some $n\in\mathbb{Z}$.

Let $f(z)=Ce^{\frac{1}{4}L(z)}$, for an appropriate constant C. We see that f is independent of the path of our integral, so it is well defined.

Let $z\in H$, and $\delta>0$ such that $B_\delta(z)\subset H$, and let $|h|<\delta$. Let p_h be the line segment from z to $z+h$. Since f is independent of paths, we have

$$\begin{aligned}f(z+h)&=Ce^{\frac{1}{4}\left(\log(z_0)+\int_\gamma\frac{g'(w)dw}{g(w)}+\int_{p_h}\frac{g'(w)dw}{g(w)}\right)}\\&=f(z)e^{\frac{1}{4}\int_{p_h}\frac{g'(w)dw}{g(w)}}.\end{aligned}$$

This gives us

$$\begin{aligned}\frac{f(z+h)-f(z)}{h} &= \frac{f(z)}{h}(e^{\frac{1}{4}\int_{p_h}\frac{g'(w)dw}{g(w)}}-1)\\ &= \frac{f(z)}{h}\sum_{n=1}^{\infty}\frac{\left(\frac{1}{4}\int_{p_h}\frac{g'(w)dw}{g(w)}\right)^n}{n!}\\ &= \frac{f(z)}{4}\frac{\left(\int_{p_h}\frac{g'(w)dw}{g(w)}\right)}{h}\sum_{n=1}^{\infty}\frac{\left(\frac{1}{4}\int_{p_h}\frac{g'(w)dw}{g(w)}\right)^{n-1}}{n!}\end{aligned}$$

so

$$\lim_{h\to 0}\frac{f(z+h)-f(z)}{h} = \frac{f(z)}{4}\frac{g'(z)}{g(z)}.$$

We see that f is differentiable on H, so it is holomorphic.

Finally, we have that for $z \in H$,

$$\begin{aligned}\frac{d}{dz}\frac{g(z)}{f(z)^4} &= \frac{-4}{4}f(z)\frac{g'(z)}{g(z)}\frac{1}{f(z)^5}g(z)+\frac{g'(z)}{f(z)^4}\\ &= 0,\end{aligned}$$

so $\frac{g(z)}{f(z)^4}$ is a constant. By choosing the appropriate C, we have that $f(z)^4 = g(z)$. □

(3) (F 2016 # 7) Show that there is a holomorphic function f on the region $|z| > 2$ such that $f(z)^4 = (z^2-1)(z^2-4)$.

Proof. Let $H = \{z : 2 < |z|\}$, and $g(z) = (z^2-1)(z^2-4)$. Let $z_0 \in H$. For all $z \in H$, let

$$L(z) = \log(g(z_0)) + \int_\gamma \frac{g'(w)dw}{g(w)}$$

where γ is a path in H from z_0 to z.

Let γ_1 and γ_2 be paths in H from z_0 to z, and let γ be the path that goes from z_0 to z on γ_1 and from z to z_0 on γ_2. If $\overline{B_2(0)}$ is not inside γ, then

$$0 = \int_\gamma \frac{g'(w)dw}{g(w)} = \int_{\gamma_1}\frac{g'(w)dw}{g(w)} - \int_{\gamma_2}\frac{g'(w)dw}{g(w)}.$$

If $\overline{B_2(0)}$ is inside of γ, then by the Residue Theorem, we have

$$\begin{aligned}\int_{\gamma_1}\frac{g'(w)dw}{g(w)} - \int_{\gamma_2}\frac{g'(w)dw}{g(w)} &= \int_\gamma \frac{g'(w)dw}{g(w)}\\ &= 2\pi i\cdot m\left(\text{Res}_{z=1}\frac{g'(w)}{g(w)}+\text{Res}_{z=-1}\frac{g'(w)}{g(w)}+\text{Res}_{z=2}\frac{g'(w)}{g(w)}+\text{Res}_{z=-2}\frac{g'(w)}{g(w)}\right)\\ &= 8\pi i\cdot m.\end{aligned}$$

So, for any two paths γ_1 and γ_2 from z_0 to z in H, we have

$$\int_{\gamma_1}\frac{g'(w)dw}{g(w)} = 8\pi i\cdot n + \int_{\gamma_2}\frac{g'(w)dw}{g(w)}$$

for some $n \in \mathbb{Z}$.

Let $f(z) = Ce^{\frac{1}{4}L(z)}$, for an appropriate constant C. We see that f is independent of the path of our integral, so it is well defined.

Let $z \in H$, and $\delta > 0$ such that $B_\delta(z) \subset H$, and let $|h| < \delta$. Let p_h be the line segment from z to $z+h$. Since f is independent of paths, we have

$$f(z+h) = Ce^{\frac{1}{4}\left(\log(z_0)+\int_\gamma \frac{g'(w)dw}{g(w)}+\int_{p_h} \frac{g'(w)dw}{g(w)}\right)}$$
$$= f(z)e^{\frac{1}{4}\int_{p_h} \frac{g'(w)dw}{g(w)}}.$$

This gives us

$$\frac{f(z+h)-f(z)}{h} = \frac{f(z)}{h}(e^{\frac{1}{4}\int_{p_h} \frac{g'(w)dw}{g(w)}} - 1)$$
$$= \frac{f(z)}{h}\sum_{n=1}^{\infty} \frac{\left(\frac{1}{4}\int_{p_h} \frac{g'(w)dw}{g(w)}\right)^n}{n!}$$
$$= \frac{f(z)}{4}\frac{\left(\int_{p_h} \frac{g'(w)dw}{g(w)}\right)}{h}\sum_{n=1}^{\infty} \frac{\left(\frac{1}{4}\int_{p_h} \frac{g'(w)dw}{g(w)}\right)^{n-1}}{n!}$$

so

$$\lim_{h\to 0} \frac{f(z+h)-f(z)}{h} = \frac{f(z)}{4}\frac{g'(z)}{g(z)}.$$

We see that f is differentiable on H, so it is holomorphic.

Finally, we have that for $z \in H$,

$$\frac{d}{dz}\frac{g(z)}{f(z)^2} = \frac{-4}{4}f(z)\frac{g'(z)}{g(z)}\frac{1}{f(z)^5}g(z) + \frac{g'(z)}{f(z)^4} = 0,$$

so $\frac{g(z)}{f(z)^4}$ is a constant. By choosing the appropriate C, we have that $f(z)^4 = g(z)$. □

(4) (S 2016 # 6) Show that there is a holomorphic function f on the region $|z| > 2$ such that $f(z)^2 = (z^2-1)(z^2-4)$.

Proof. Let $H = \{z : 2 < |z|\}$, and $g(z) = (z^2-1)(z^2-4)$. Let $z_0 \in H$. For all $z \in H$, let

$$L(z) = \log(g(z_0)) + \int_\gamma \frac{g'(w)dw}{g(w)}$$

where γ is a path in H from z_0 to z.

Let γ_1 and γ_2 be paths in H from z_0 to z, and let γ be the path that goes from z_0 to z on γ_1 and from z to z_0 on γ_2. If $\overline{B_2(0)}$ is not inside γ, then

$$0 = \int_\gamma \frac{g'(w)dw}{g(w)} = \int_{\gamma_1} \frac{g'(w)dw}{g(w)} - \int_{\gamma_2} \frac{g'(w)dw}{g(w)}.$$

If $\overline{B_2(0)}$ is inside of γ, then by the Residue Theorem, we have

$$\int_{\gamma_1} \frac{g'(w)dw}{g(w)} - \int_{\gamma_2} \frac{g'(w)dw}{g(w)} = \int_\gamma \frac{g'(w)dw}{g(w)}$$
$$= 2\pi i \cdot m\left(\text{Res}_{z=1}\frac{g'(w)}{g(w)} + \text{Res}_{z=-1}\frac{g'(w)}{g(w)} + \text{Res}_{z=2}\frac{g'(w)}{g(w)} + \text{Res}_{z=-2}\frac{g'(w)}{g(w)}\right)$$
$$= 8\pi i \cdot m.$$

So, for any two paths γ_1 and γ_2 from z_0 to z in H, we have

$$\int_{\gamma_1} \frac{g'(w)dw}{g(w)} = 8\pi i \cdot n + \int_{\gamma_2} \frac{g'(w)dw}{g(w)}$$

for some $n \in \mathbb{Z}$.

Let $f(z) = Ce^{\frac{1}{2}L(z)}$, for an appropriate constant C. We see that f is independent of the path of our integral, so it is well defined.

Let $z \in H$, and $\delta > 0$ such that $B_\delta(z) \subset H$, and let $|h| < \delta$. Let p_h be the line segment from z to $z+h$. Since f is independent of paths, we have

$$\begin{aligned} f(z+h) &= Ce^{\frac{1}{2}\left(\log(z_0)+\int_\gamma \frac{g'(w)dw}{g(w)}+\int_{p_h} \frac{g'(w)dw}{g(w)}\right)} \\ &= f(z)e^{\frac{1}{2}\int_{p_h} \frac{g'(w)dw}{g(w)}}. \end{aligned}$$

This gives us

$$\begin{aligned} \frac{f(z+h)-f(z)}{h} &= \frac{f(z)}{h}(e^{\frac{1}{2}\int_{p_h} \frac{g'(w)dw}{g(w)}} - 1) \\ &= \frac{f(z)}{h}\sum_{n=1}^{\infty} \frac{\left(\frac{1}{2}\int_{p_h} \frac{g'(w)dw}{g(w)}\right)^n}{n!} \\ &= \frac{f(z)}{2}\frac{\left(\int_{p_h} \frac{g'(w)dw}{g(w)}\right)}{h}\sum_{n=1}^{\infty} \frac{\left(\frac{1}{2}\int_{p_h} \frac{g'(w)dw}{g(w)}\right)^{n-1}}{n!} \end{aligned}$$

so

$$\lim_{h\to 0} \frac{f(z+h)-f(z)}{h} = \frac{f(z)}{2}\frac{g'(z)}{g(z)}.$$

We see that f is differentiable on H, so it is holomorphic.

Finally, we have that for $z \in H$,

$$\frac{d}{dz}\frac{g(z)}{f(z)^2} = \frac{-2}{2}f(z)\frac{g'(z)}{g(z)}\frac{1}{f(z)^3}g(z) + \frac{g'(z)}{f(z)^2} = 0,$$

so $\frac{g(z)}{f(z)^2}$ is a constant. By choosing the appropriate C, we have that $f(z)^2 = g(z)$. □

(5) (S 2015 # 8) Show that there is a holomorphic function $f(z)$ on a neighborhood of 0 so that $f(z)^2 = \frac{\sin z}{z}$, and determine radius of convergence of the power series at 0.

Proof. Let $H = \{z : |z| < \pi\}$, and $g(z) = \frac{\sin(z)}{z}$. We note that the singularity at $z = 0$ is removable, with $g(0) = 1$, as

$$\frac{\sin(z)}{z} = \sum_{n=0}^{\infty}(-1)^n \frac{z^{2n}}{(2n+1)!} = g(z).$$

This sum converges everywhere, so g is an entire function.

Let $z_0 \in H$. For all $z \in H$, let

$$L(z) = \log(g(z_0)) + \int_\gamma \frac{g'(w)dw}{g(w)}$$

where γ is a path in H from z_0 to z.

Since there are no zeros of g in H, we have that for any two paths γ_1 and γ_2 from z_0 to z in H,

$$\int_{\gamma_1} \frac{g'(w)dw}{g(w)} = \int_{\gamma_2} \frac{g'(w)dw}{g(w)}.$$

Let $f(z) = Ce^{\frac{1}{2}L(z)}$, for an appropriate constant C. We see that f is independent of the path of our integral, so it is well defined.

Let $z \in H$, and $\delta > 0$ such that $B_\delta(z) \subset H$, and let $|h| < \delta$. Let p_h be the line segment from z to $z+h$. Since f is independent of paths, we have

$$\begin{aligned} f(z+h) &= Ce^{\frac{1}{2}\left(\log(z_0)+\int_\gamma \frac{g'(w)dw}{g(w)}+\int_{p_h}\frac{g'(w)dw}{g(w)}\right)} \\ &= f(z)e^{\frac{1}{2}\int_{p_h}\frac{g'(w)dw}{g(w)}}. \end{aligned}$$

This gives us

$$\begin{aligned} \frac{f(z+h)-f(z)}{h} &= \frac{f(z)}{h}(e^{\frac{1}{2}\int_{p_h}\frac{g'(w)dw}{g(w)}}-1) \\ &= \frac{f(z)}{h}\sum_{n=1}^{\infty}\frac{\left(\frac{1}{2}\int_{p_h}\frac{g'(w)dw}{g(w)}\right)^n}{n!} \\ &= \frac{f(z)}{2}\frac{\left(\int_{p_h}\frac{g'(w)dw}{g(w)}\right)}{h}\sum_{n=1}^{\infty}\frac{\left(\frac{1}{2}\int_{p_h}\frac{g'(w)dw}{g(w)}\right)^{n-1}}{n!} \end{aligned}$$

so

$$\lim_{h\to 0}\frac{f(z+h)-f(z)}{h} = \frac{f(z)}{2}\frac{g'(z)}{g(z)}.$$

We see that f is differentiable on H, so it is holomorphic.

Finally, we have that for $z \in H$,

$$\frac{d}{dz}\frac{g(z)}{f(z)^2} = \frac{-2}{2}f(z)\frac{g'(z)}{g(z)}\frac{1}{f(z)^3}g(z) + \frac{g'(z)}{f(z)^2} = 0,$$

so $\frac{g(z)}{f(z)^2}$ is a constant. By choosing the appropriate C, we have that $f(z)^2 = g(z)$.

Since any function agreeing with $f(z)$ has a branch point at $z = \pi$, we see that the radius of convergence of power series at 0 of f is π. □

(6) (F 2014 # 9) Show that there is a holomorphic function $f(z)$ on a neighborhood of 0 so that $f(z)^2 = \frac{e^z-1}{z}$. What is the radius of convergence of the power series at 0 for this function?

Proof. Let $H = \{z : |z| < 2\pi\}$, and $g(z) = \frac{e^z-1}{z}$. We note that the singularity at $z = 0$ is removable, with $g(0) = 1$, as

$$\frac{e^z-1}{z} = \sum_{n=1}^{\infty}\frac{z^{n-1}}{n!} = g(z).$$

This sum converges everywhere, so g is an entire function.

Let $z_0 \in H$. For all $z \in H$, let

$$L(z) = \log(g(z_0)) + \int_\gamma \frac{g'(w)dw}{g(w)}$$

where γ is a path in H from z_0 to z.

Since there are no zeros of g in H, we have that for any two paths γ_1 and γ_2 from z_0 to z in H, we have

$$\int_{\gamma_1}\frac{g'(w)dw}{g(w)} = \int_{\gamma_2}\frac{g'(w)dw}{g(w)}.$$

Let $f(z) = Ce^{\frac{1}{2}L(z)}$, for an appropriate constant C. We see that f is independent of the path of our integral, so it is well defined.

Let $z \in H$, and $\delta > 0$ such that $B_\delta(z) \subset H$, and let $|h| < \delta$. Let p_h be the line segment from z to $z+h$. Since f is independent of paths, we have

$$\begin{aligned} f(z+h) &= Ce^{\frac{1}{2}\left(\log(z_0)+\int_\gamma \frac{g'(w)dw}{g(w)}+\int_{p_h}\frac{g'(w)dw}{g(w)}\right)} \\ &= f(z)e^{\frac{1}{2}\int_{p_h}\frac{g'(w)dw}{g(w)}}. \end{aligned}$$

This gives us

$$\begin{aligned}\frac{f(z+h)-f(z)}{h} &= \frac{f(z)}{h}(e^{\frac{1}{2}\int_{p_h}\frac{g'(w)dw}{g(w)}}-1)\\ &= \frac{f(z)}{h}\sum_{n=1}^{\infty}\frac{\left(\frac{1}{2}\int_{p_h}\frac{g'(w)dw}{g(w)}\right)^n}{n!}\\ &= \frac{f(z)}{2}\frac{\left(\int_{p_h}\frac{g'(w)dw}{g(w)}\right)}{h}\sum_{n=1}^{\infty}\frac{\left(\frac{1}{2}\int_{p_h}\frac{g'(w)dw}{g(w)}\right)^{n-1}}{n!}\end{aligned}$$

so

$$\lim_{h\to 0}\frac{f(z+h)-f(z)}{h} = \frac{f(z)}{2}\frac{g'(z)}{g(z)}.$$

We see that f is differentiable on H, so it is holomorphic.

Finally, we have that for $z \in H$,

$$\frac{d}{dz}\frac{g(z)}{f(z)^2} = \frac{-2}{2}f(z)\frac{g'(z)}{g(z)}\frac{1}{f(z)^3}g(z) + \frac{g'(z)}{f(z)^2} = 0,$$

so $\frac{g(z)}{f(z)^2}$ is a constant. By choosing the appropriate C, we have that $f(z)^2 = g(z)$.

Since any function agreeing with $f(z)$ has a branch point at $z = 2\pi i$, we see that the radius of convergence of power series at 0 of f is 2π. □

12. Trigonometric Formulas

(1) (F 2017 # 7) Show that $\frac{\sin(\pi z)}{\pi z} = \prod_{n\geq 1}\left(1-\frac{z^2}{n^2}\right)$.

Proof. Since

$$\frac{\pi^2}{\sin^2(\pi z)} = \sum_{n\in\mathbb{Z}}\frac{1}{(z-n)^2} = \frac{1}{z^2} + \sum_{n\in\mathbb{N}}\left(\frac{1}{(z+n)^2}+\frac{1}{(z-n)^2}\right),$$

we have that, for some constant C,

$$-\pi\cot(\pi z) = C - \frac{1}{z} - \sum_{n\in\mathbb{N}}\left(\frac{1}{z+n}+\frac{1}{z-n}\right),$$

as the first equation is the derivative of the second.

Since

$$\frac{1}{z+n}+\frac{1}{z-n} = \frac{2z}{z^2-n^2},$$

we know that the sum converges uniformly and absolutely, and the summands are odd functions of z. Since $\cot(\pi z)$ is an odd function, we have that $C = 0$, so

$$\pi\cot(\pi z) = \frac{1}{z} + \sum_{n\in\mathbb{N}}\left(\frac{1}{z+n}+\frac{1}{z-n}\right).$$

We have that

$$\frac{d}{dz}\log(\sin(\pi z)) = \pi\cot(\pi z),$$

and for any constant C,

$$\frac{d}{dz}\left(C + \log(z) + \sum_{n\in\mathbb{N}}\left(\log\left(1-\frac{z}{n}\right)+\log\left(1+\frac{z}{n}\right)\right)\right) = \frac{1}{z} + \sum_{n\in\mathbb{N}}\left(\frac{1}{z+n}+\frac{1}{z-n}\right).$$

Thus

$$\sin(\pi z) = e^C z \prod_{n\in\mathbb{N}} \left(1 - \frac{z^2}{n^2}\right).$$

Considering the power series of $\sin(\pi z)$ centered at 0, we know that $e^C = \pi$, so

$$\frac{\sin(\pi z)}{\pi z} = \prod_{n\in\mathbb{N}} \left(1 - \frac{z^2}{n^2}\right).$$

□

(2) (S 2017 # 7) Show that $\frac{\pi^2}{\sin^2(\pi z)} = \sum_{n\in\mathbb{Z}} \frac{1}{(z-n)^2}$.

Proof. Let $f(z) = \frac{\pi^2}{\sin^2(\pi z)}$ and $g(z) = \sum_{n\in\mathbb{Z}} \frac{1}{(z-n)^2}$. g converges for $z \notin \mathbb{Z}$, so it is holomorphic away from the poles. We note that both functions have double poles at the integers.

The Laurent expansion of g near $n \in \mathbb{Z}$ takes the form

$$\frac{1}{(z-n)^2} + \text{ holomorphic.}$$

The Laurent expansion of f near 0 is given by

$$\begin{aligned}
\frac{\pi^2}{\sin^2(\pi z)} &= \frac{\pi^2}{\left(\pi z - \frac{(\pi z)^3}{3!} + \cdots\right)^2} \\
&= \frac{1}{z^2}\frac{1}{\left(1 - \frac{(\pi z)^2}{3!} + \cdots\right)^2} \\
&= \frac{1}{z^2}\frac{1}{\left(1 - \frac{2(\pi z)^2}{3!} + \cdots\right)} \\
&= \frac{1}{z^2}\left(1 - \frac{\left(-\frac{2(\pi z)^2}{3!} + \cdots\right)}{\left(1 - \frac{2(\pi z)^2}{3!} + \cdots\right)}\right) \\
&= \frac{1}{z^2}\left(1 + \frac{2(\pi z)^2}{3!} + \cdots\right) \\
&= \frac{1}{z^2} + \text{ holomorphic.}
\end{aligned}$$

Since the periodicity of f then gives us that for each $n \in \mathbb{Z}$, the Laurent expansion of f near n takes the form

$$\frac{1}{(z-n)^2} + \text{ holomorphic.}$$

Thus, the poles of f and g cancel, meaning that

$$h(z) = f(z) - g(z)$$

is an entire function.

Since h is entire, it is bounded on $\{x + iy : 0 \le x \le 1, |y| \le r\}$ for any $r \ge 0$. Since $h(z) = h(z+1)$ for all z, we have that h is bounded on $\{x + iy : |y| \le r\}$ for any $r \ge 0$. Since

$$\lim_{|y|\to\infty} |f(x+iy)| = \lim_{|y|\to\infty} \left|\frac{-4\pi^2}{e^{2\pi(xi-y)} + e^{-2\pi(xi-y)} - 2}\right| = 0$$

and

$$\lim_{|y|\to\infty} |g(x+iy)| \le \lim_{|y|\to\infty} \sum_{n\in\mathbb{Z}} \frac{1}{|x-n+iy|^2} = 0,$$

we see that

$$\lim_{|y|\to\infty} |h(x+iy)| = 0.$$

Thus, h is entire and bounded, and by Liouville's Theorem, this means that h is constant. Since $\lim_{y\to\infty} h(x+iy) = 0$, this means that h is identically zero, so $f = g$.

□

(3) (F 2016 # 8) Show that $\frac{\pi^2}{\sin^2(\pi z)} = \sum_{n\in\mathbb{Z}} \frac{1}{(z-n)^2}$.

Proof. See above. □

(4) (F 2015 # 8) Show that $\frac{\pi^2}{\sin^2(\pi z)} = \sum_{n\in\mathbb{Z}} \frac{1}{(z-n)^2}$.

Proof. See above. □

13. Harmonic Functions

Theorem 13 (Parseval's Identity)**.**

$$\int_{-1}^{1} |f(x)|^2 dx = \sum_{n\in\mathbb{Z}} |c_n|^2$$

where

$$c_n = \int_{-1}^{1} f(x) e^{2\pi i n x} dx.$$

Proposition 4 (Poisson's formula)**.** *For u harmonic on a neighborhood of the closed unit disk $|z| \le R$, u is expressible in terms of its boundary values on $|z| = R$ by*

$$u(z) = \frac{1}{2\pi R}\int_0^{2\pi} f(Re^{it}) \frac{R^2 - |z|^2}{|z - Re^{it}|^2} dt \ \ (|z| < R).$$

(1) (S 2016 # 7) Describe all complex-valued harmonic functions on the punctured disk $0 < |z| < 1$.

Proof. Let $u(z)$ be a harmonic function on $0 < |z| < 1$. We can express u in polar coordinates as a Fourier series in θ with coefficients $c_n(r)$:

$$u(re^{i\theta}) = \sum_{n\in\mathbb{Z}} c_n(r) e^{in\theta}.$$

Applying the Laplacian in polar coordinates to the Fourier series and differentiating termwise,

$$\Delta u = \sum_{n\in\mathbb{Z}} \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right) c_n(r) e^{in\theta} = \sum_{n\in\mathbb{Z}} \left(c_n''(r) + \frac{1}{r}c_n'(r) - \frac{n^2}{r^2}c_n(r)\right) e^{in\theta} = 0 = \sum_{n\in\mathbb{Z}} 0\cdot e^{in\theta},$$

since u is harmonic. By uniqueness of Fourier expansions,

$$c_n''(r) + \frac{1}{r}c_n'(r) - \frac{n^2}{r^2}c_n(r) = 0$$

for all $n \in \mathbb{Z}$. This ordinary differential equation is of Euler type, so we take $c_n = r^m$ and solve for m:

$$(r^m)'' + \frac{1}{r}(r^m)' - \frac{n^2}{r^2} r^m = 0,$$

which implies

$$m(m-1) + m - n^2 = 0,$$

so $m = \pm n$. For $n \neq 0$, the solutions are r^n and r^{-n}, and since the root $m = 0$ is doubled, the solutions for $n = 0$ are $r^0 = 1$ and $\log(r)$. Taking linear combinations of these solutions and changing back to z, $\overline{z}$ coordinate,

$$\begin{aligned} u(z) &= a_0 + b_0 \log(r) + \sum_{n\in\mathbb{Z}\setminus\{0\}} (a_n r^n + b_n r^{-n}) e^{in\theta} \\ &= a_0 + b_0 \log(|z|) + \sum_{n\in\mathbb{Z}\setminus\{0\}} (a_n z^n + b_n \overline{z}^{-n}). \end{aligned}$$

Since harmonic functions are continuous and thus L^2 on $0 \leq |z| \leq 1$, their Fourier coefficients must have a finite sum of squares of absolute values for $0 < |z| < 1$ by Parseval's Identity. Thus, for a_n and b_n with sufficiently rapid decay, $u(z) = a_0 + b_0 \log(|z|) + \sum_{n\in\mathbb{Z}\setminus\{0\}}(a_n z^n + b_n \overline{z}^{-n})$ converges nicely to a harmonic function on the punctured disk.

□

(2) (F 2015 # 9) Let f be a harmonic function on $\mathbb{C}$, such that $|f(z)| \leq \sqrt{1 + |z|}$. Show f is constant.

Proof. Let f be harmonic on $\mathbb{C}$, such that $|f(z)| \leq \sqrt{1 + |z|}$ for all $z \in \mathbb{C}$. By Poisson's formula for harmonic functions, integrating counterclockwise over the circle of radius $R > 0$ centered at 0

$$f(z) = \frac{1}{2\pi R} \int_0^{2\pi} f(Re^{it}) \frac{R^2 - |z|^2}{|z - Re^{it}|^2} dt.$$

For $|z| < \frac{R}{2}$

$$\begin{aligned} |f(z) - f(0)| &= \left| \frac{1}{2\pi R} \int_0^{2\pi} f(Re^{it}) \left(\frac{R^2 - |z|^2}{|z - Re^{it}|^2} - 1 \right) dt \right| \\ &\leq \frac{1}{2\pi R} \int_0^{2\pi} f(Re^{it}) \frac{R^2 - |z|^2}{|z - Re^{it}|^2} dt \\ &\leq \frac{1}{2\pi R} \int_0^{2\pi} \frac{\sqrt{1+R}R^2}{(R/2)^2} dt \\ &= \frac{4\sqrt{1+R}}{R}. \end{aligned}$$

This goes to 0 as R goes to ∞, so $f(z) = f(0)$ for all $z \in \mathbb{C}$. Thus, f is indeed constant. □

(3) (S 2015 # 9) Describe all complex-valued harmonic functions on the annulus $1 < |z| < 2$ which extend continuously to the circle $|z| = 2$ and take value 0 on the circle.

Proof. Let $u(z)$ be a harmonic function on $1 < |z| < 2$. We can express u in polar coordinates as a Fourier series in θ with coefficients $c_n(r)$:

$$u(re^{i\theta}) = \sum_{n\in\mathbb{Z}} c_n(r) e^{in\theta}.$$

Applying the Laplacian in polar coordinates to the Fourier series and differentiating termwise,

$$\Delta u = \sum_{n\in\mathbb{Z}} \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} \right) c_n(r) e^{in\theta} = \sum_{n\in\mathbb{Z}} \left(c_n''(r) + \frac{1}{r} c_n'(r) - \frac{n^2}{r^2} c_n(r) \right) e^{in\theta} = 0 = \sum_{n\in\mathbb{Z}} 0 \cdot e^{in\theta},$$

since u is harmonic. By uniqueness of Fourier expansions,

$$c_n''(r) + \frac{1}{r} c_n'(r) - \frac{n^2}{r^2} c_n(r) = 0$$

for all $n \in \mathbb{Z}$. This ordinary differential equation is of Euler type, so we take $c_n = r^m$ and solve for m:

$$(r^m)'' + \frac{1}{r}(r^m)' - \frac{n^2}{r^2} r^m = 0,$$

which implies

$$m(m-1)+m-n^2=0,$$

so $m=\pm n$. For $n\neq 0$, the solutions are r^n and r^{-n}, and since the root $m=0$ is doubled, the solutions for $n=0$ are $r^0=1$ and $\log(r)$. Taking linear combinations of these solutions and changing back to z, $\overline{z}$ coordinate,

$$\begin{aligned}u(z) &= a_0+b_0\log(r)+\sum_{n\in\mathbb{Z}\setminus\{0\}}(a_nr^n+b_nr^{-n})e^{in\theta}\\ &= a_0+b_0\log(|z|)+\sum_{n\in\mathbb{Z}\setminus\{0\}}(a_nz^n+b_n\overline{z}^{-n}).\end{aligned}$$

Our function must satisfy that $u=0$ on $|z|=r=2$:

$$u(2e^{i\theta})=a_0+b_0\log(2)+\sum_{n\in\mathbb{Z}\setminus\{0\}}(a_n2^n+b_n2^{-n})e^{in\theta}=0=\sum_{n\in\mathbb{Z}\setminus\{0\}}0\cdot e^{in\theta}.$$

By uniqueness of Fourier expansions, $a_0+b_0\log(2)=a_n2^n+b_n2^{-n}=0$. Thus

$$u(z)=u(re^{i\theta})=b_0\log(\frac{r}{2})+\sum_{n\in\mathbb{Z}\setminus\{0\}}a_n(r^n-4^nr^{-n})e^{in\theta}=b_0\log(\frac{|z|}{2})+\sum_{n\in\mathbb{Z}\setminus\{0\}}a_n(z^n-4^n\overline{z}^{-n}).$$

Since harmonic functions are continuous and thus L^2 on any circle of radius $1<r\leq 2$, their Fourier coefficients must have a finite sum of squares of absolute values for $1<|z|<2$ by Parseval's Identity. Thus, for a_n with sufficiently rapid decay, $u(z)=b_0\log|\frac{z}{2}|+\sum_{n\in\mathbb{Z}\setminus\{0\}}a_n(z^n-4^n\overline{z}^{-n})$ converges nicely to a harmonic function on the annulus satisfying the given boundary conditions.

□

(4) (F 2014 # 8) Describe all complex-valued harmonic functions on the punctured disk $0<|z|<1$ which extend continuously to the circle $|z|=1$ and take value 0 on that circle.

Proof. Let $u(z)$ be a harmonic function on $0<|z|<1$. We can express u in polar coordinates as a Fourier series in θ with coefficients $c_n(r)$:

$$u(re^{i\theta})=\sum_{n\in\mathbb{Z}}c_n(r)e^{in\theta}.$$

Applying the Laplacian in polar coordinates to the Fourier series and differentiating termwise,

$$\Delta u=\sum_{n\in\mathbb{Z}}\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)c_n(r)e^{in\theta}=\sum_{n\in\mathbb{Z}}\left(c_n''(r)+\frac{1}{r}c_n'(r)-\frac{n^2}{r^2}c_n(r)\right)e^{in\theta}=0=\sum_{n\in\mathbb{Z}}0\cdot e^{in\theta},$$

since u is harmonic. By uniqueness of Fourier expansions,

$$c_n''(r)+\frac{1}{r}c_n'(r)-\frac{n^2}{r^2}c_n(r)=0$$

for all $n\in\mathbb{Z}$. This ordinary differential equation is of Euler type, so we take $c_n=r^m$ and solve for m:

$$(r^m)''+\frac{1}{r}(r^m)'-\frac{n^2}{r^2}r^m=0,$$

which implies

$$m(m-1)+m-n^2=0,$$

so $m=\pm n$. For $n\neq 0$, the solutions are r^n and r^{-n}, and since the root $m=0$ is doubled, the solutions for $n=0$ are $r^0=1$ and $\log(r)$. Taking linear combinations of these solutions and changing back to z, $\overline{z}$

coordinate,

$$u(z) = a_0 + b_0 \log(r) + \sum_{n\in\mathbb{Z}\setminus\{0\}} (a_n r^n + b_n r^{-n})e^{in\theta}$$
$$= a_0 + b_0 \log(|z|) + \sum_{n\in\mathbb{Z}\setminus\{0\}} (a_n z^n + b_n \overline{z}^{-n}).$$

Our function must satisfy that $u = 0$ on $|z| = r = 1$:

$$u(2e^{i\theta}) = a_0 + \sum_{n\in\mathbb{Z}\setminus\{0\}} (a_n + b_n)e^{in\theta} = 0 = \sum_{n\in\mathbb{Z}\setminus\{0\}} 0 \cdot e^{in\theta}.$$

By uniqueness of Fourier expansions, $a_0 = a_n + b_n = 0$. Thus

$$u(z) = u(re^{i\theta}) = b_0 \log(r) + \sum_{n\in\mathbb{Z}\setminus\{0\}} a_n(r^n - r^{-n})e^{in\theta} = b_0 \log(|z|) + \sum_{n\in\mathbb{Z}\setminus\{0\}} a_n(z^n - \overline{z}^{-n}).$$

Since harmonic functions are continuous and thus L^2 on $0 \leq |z| \leq 1$, their Fourier coefficients must have a finite sum of squares of absolute values for $0 < |z| < 1$ by Parseval's Identity. Thus, for a_n with sufficiently rapid decay, $u(z) = b_0 \log(|z|) + \sum_{n\in\mathbb{Z}\setminus\{0\}} a_n(z^n - \overline{z}^{-n})$ converges nicely to a harmonic function on the punctured disk satisfying the given boundary conditions.

□

14. Inverse Function

Theorem 14 (Inverse Function Theorem). *For f holomorphic on a neighborhood U of z_0 and $f'(z_0) \neq 0$, there is a holomorphic inverse function, g, of f on a neighborhood of $f(z_0)$.*

(1) (S 2018 # 7) Try to define w locally as a holomorphic function of z defined by the relation $w^5 - 5zw + 1 = 0$. What are the branch points?

Proof. Let $f(w) = \frac{w^5+1}{5w}$ for $w \neq 0$. This function is holomorphic on $\mathbb{C} \setminus \{0\}$, and

$$f'(w) = \frac{4}{5}w^3 - \frac{1}{5w^2}$$

is nonzero for $w^5 \neq \frac{1}{4}$. Let $w_0 \in D = \mathbb{C} \setminus \{0, 4^{-1/5}\omega, 4^{-1/5}\omega^2, 4^{-1/5}\omega^3, 4^{-1/5}\omega^4, 4^{-1/5}\}$, with $\omega = e^{2\pi i/5}$. By the Inverse Function Theorem, we know that on some neighborhood of $f(w_0)$, there exists some inverse function g.

Since there exists a local holomorphic inverse function of f on $f(D)$, g can have no branch points there.

For $w_0 \in \{4^{-1/5}\omega, 4^{-1/5}\omega^2, 4^{-1/5}\omega^3, 4^{-1/5}\omega^4, 4^{-1/5}\}$, since $f'(w_0) = 0$, we can write $f(w) = c + (w - w_0)^n h(w)$ for some constant c and holomorphic h with $h(w_0) \neq 0$, meaning that $n > 1$.

If w traverses a small circle near w_0, then $z = f(w)$ goes n times around $f(w_0)$. Thus, if g is the inverse of f, then w_0 is a branch point of g. □